The Workload Distribution in a MAP/G/1 Queue with Disasters

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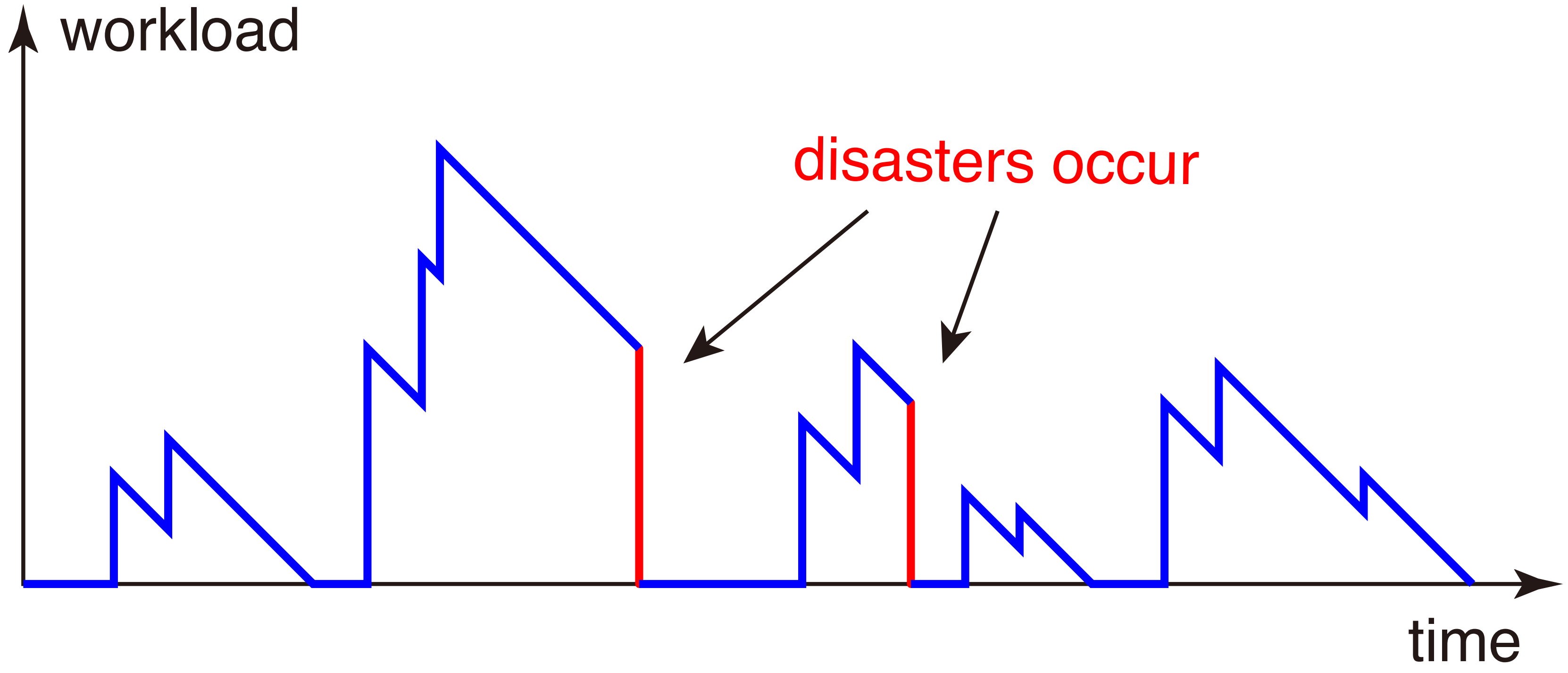
Queueing Model with Disasters

- When disasters occur, all work in system is removed



Related Works

- The BMAP/SM/1 queue with disasters [Dudin and Nishimura (1999)]
  - Customer arrival, disaster occurrence, and service times are governed by respective independent underlying Markov chains
- The BMAP/G/1 queue with disasters [Shin (2004)]
  - Customer arrival and disaster occurrence are governed by a common underlying Markov chain
  - Service times are i.i.d.

Model Considered in This Work

- Customer arrival and disaster occurrence

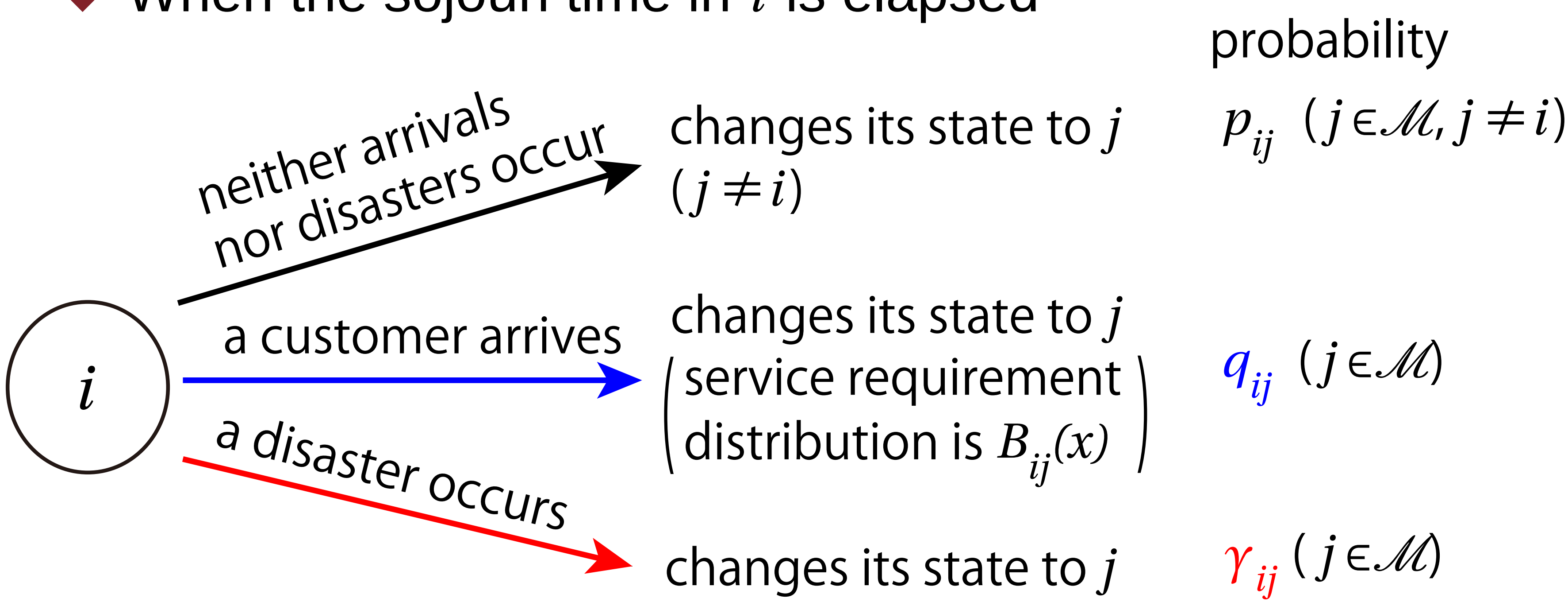
  are governed by a common underlying Markov chain

- Service requirement distributions of customers

  depend on the states of the underlying Markov chain immediately before and after arrivals

Underlying Markov Chain

- An irreducible continuous-time Markov chain with finite state space $\mathscr{M} = \{1, 2, \ldots, M\}$
- The underlying Markov chain stays in state i ($i \in \mathscr{M}$) for an exponential interval of time with mean $1/\sigma_i$
  - When the sojoun time in i is elapsed



Representation with Matrices (1)

- We introduce $M \times M$ matrices $\boldsymbol{C}$, $\boldsymbol{D}(x)$, and $\boldsymbol{\Gamma}$

$$[\boldsymbol{C}]_{i,j} = \begin{cases} \sigma_i p_{i,j}, & i \neq j \\ -\sigma_i, & i = j \end{cases}$$

Transition rate from i to j when neither customer arrivals nor disasters occur

σ_i denotes the transition rate from i

$$[\boldsymbol{D}(x)]_{i,j} = \sigma_i q_{i,j} B_{i,j}(x)$$

Transition rate from i to j when a customer arrives and the amount of service requirement is not greater than x

$$[\boldsymbol{\Gamma}]_{i,j} = \sigma_i \gamma_{i,j}$$

Transition rate from i to j when a disaster occurs

Representation with Matrices (2)

- We define $\boldsymbol{D}^*(s)$ and $\boldsymbol{D}$ as
  - $\boldsymbol{D}^*(s) = \int_0^\infty \exp(-sx)\, d\boldsymbol{D}(x), \qquad \boldsymbol{D} = \lim_{x\to\infty} \boldsymbol{D}(x) = \lim_{s\to 0+} \boldsymbol{D}^*(s)$
- $\boldsymbol{C} + \boldsymbol{D} + \boldsymbol{\Gamma}$: The infinitesimal generator of the underlying Markov chain
- We assume that $\boldsymbol{D} \neq \boldsymbol{0}$ and $\boldsymbol{\Gamma} \neq \boldsymbol{0}$
- The system becomes empty when a disaster occurs
  - ➡ $\boldsymbol{\Gamma} \neq \boldsymbol{0}$ and the irreducibility of the underlying Markov chain ensure the existance of the steady state

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Outline of the Analysis

- We consider the first passage time to the idle state, given
  - the initial workload
  - the initial state of the underlying Markov chain

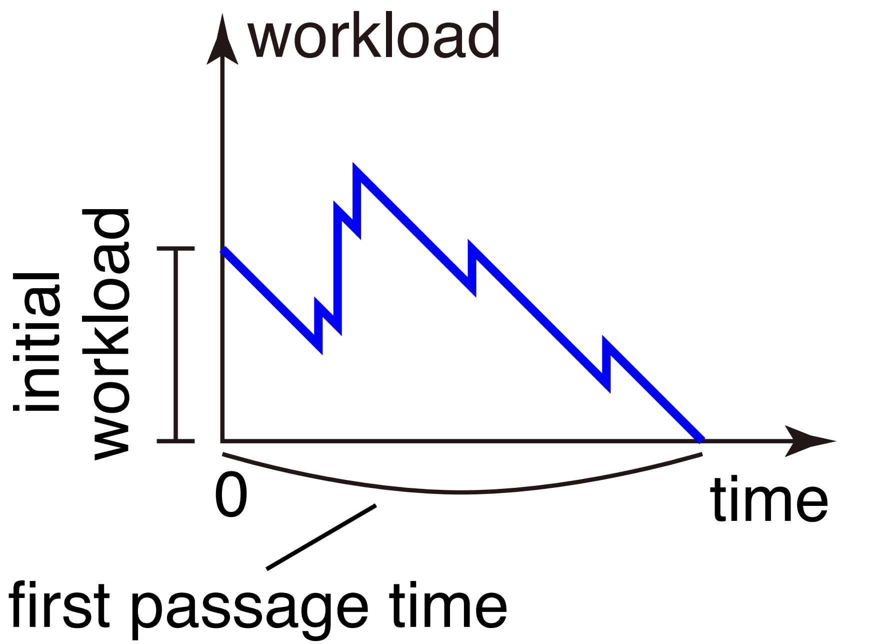
With results on the first passage time,

- Two different representations of the LST of
  the stationary distribution of work in system
  - can be derived in a similar way to that
    for an ordinary MAP/G/1 queue
    - [Takine and Hasegawa (1994)], [Takine (2002)]

First Passage Time to the Idle State

First Passage Time to the Idle State

- Classification of the first passage process
  - No disasters occur in the first passage time
  - A disaster occurs in the first passage time



LST of the First Passage Time

U_t : The amount of work in system at time t
S_t : The state of the underlying Markov chain at time t
T_{E} : The time the system first becomes empty after time 0

- $\boldsymbol{P}_{\mathrm{N}}(t \mid x)$: An $M \times M$ matrix whose (i, j)th element is given by

$$\Pr\left(T_{\mathrm{E}} \le t,\ S_{T_{\mathrm{E}}} = j,\ \text{no disasters occur} \,\middle|\, U_0 = x,\ S_0 = i\right)$$

  - $\boldsymbol{P}^*_{\mathrm{N}}(s \mid x)$: The LST of $\boldsymbol{P}_{\mathrm{N}}(t \mid x)$ with respect to t

- $\boldsymbol{P}_{\mathrm{D}}(t \mid x)$: An $M \times M$ matrix whose (i, j)th element is given by

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  - $\boldsymbol{P}^*_{\mathrm{D}}(s \mid x)$: The LST of $\boldsymbol{P}_{\mathrm{D}}(t \mid x)$ with respect to t

Properties of the First Passage Time

- No disasters occur in the first passage time
  - $\boldsymbol{P}_{\mathrm{N}}^{*}(s \mid x+y) = \boldsymbol{P}_{\mathrm{N}}^{*}(s \mid x) \cdot \boldsymbol{P}_{\mathrm{N}}^{*}(s \mid y)$
- A disaster occurs in the first passage time
  - $\boldsymbol{P}_{\mathrm{D}}^{*}(s \mid x+y) = \boldsymbol{P}_{\mathrm{D}}^{*}(s \mid x) + \boldsymbol{P}_{\mathrm{N}}^{*}(s \mid x) \cdot \boldsymbol{P}_{\mathrm{D}}^{*}(s \mid y)$

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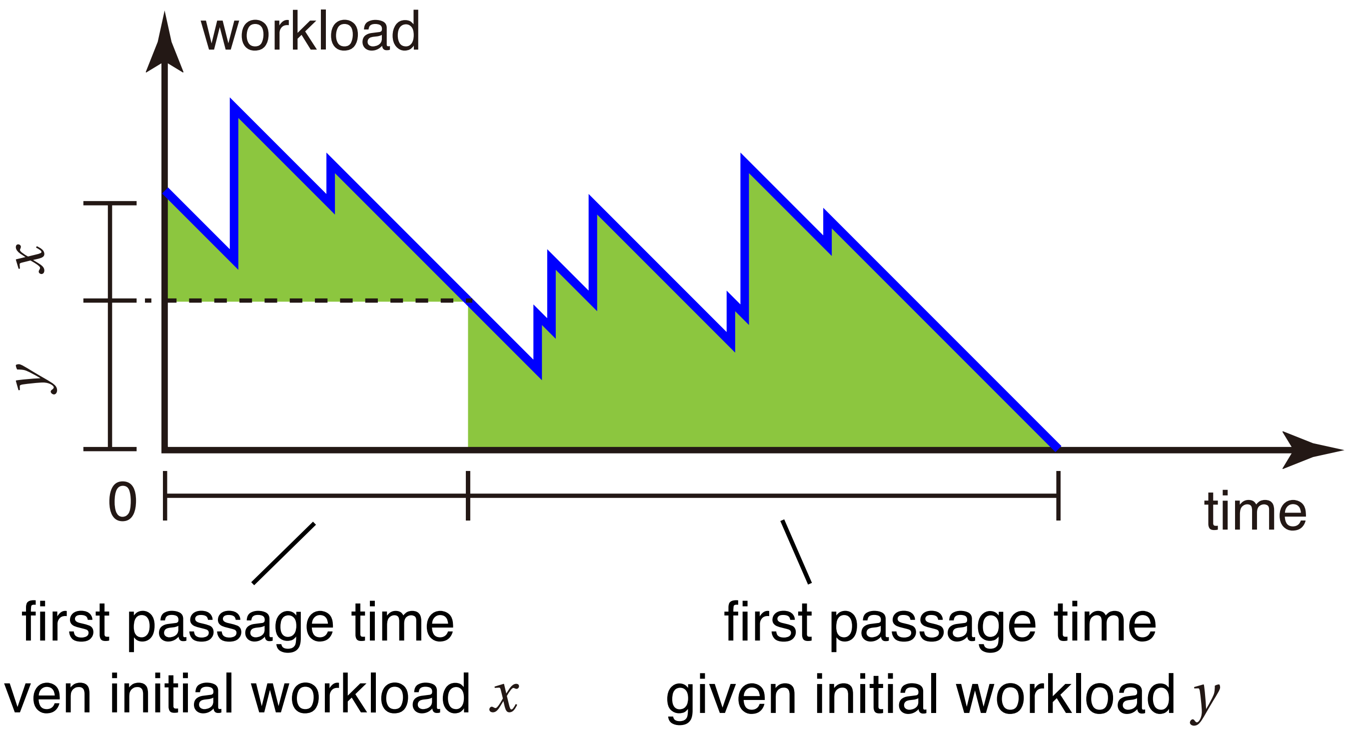

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Properties of the First Passage Time

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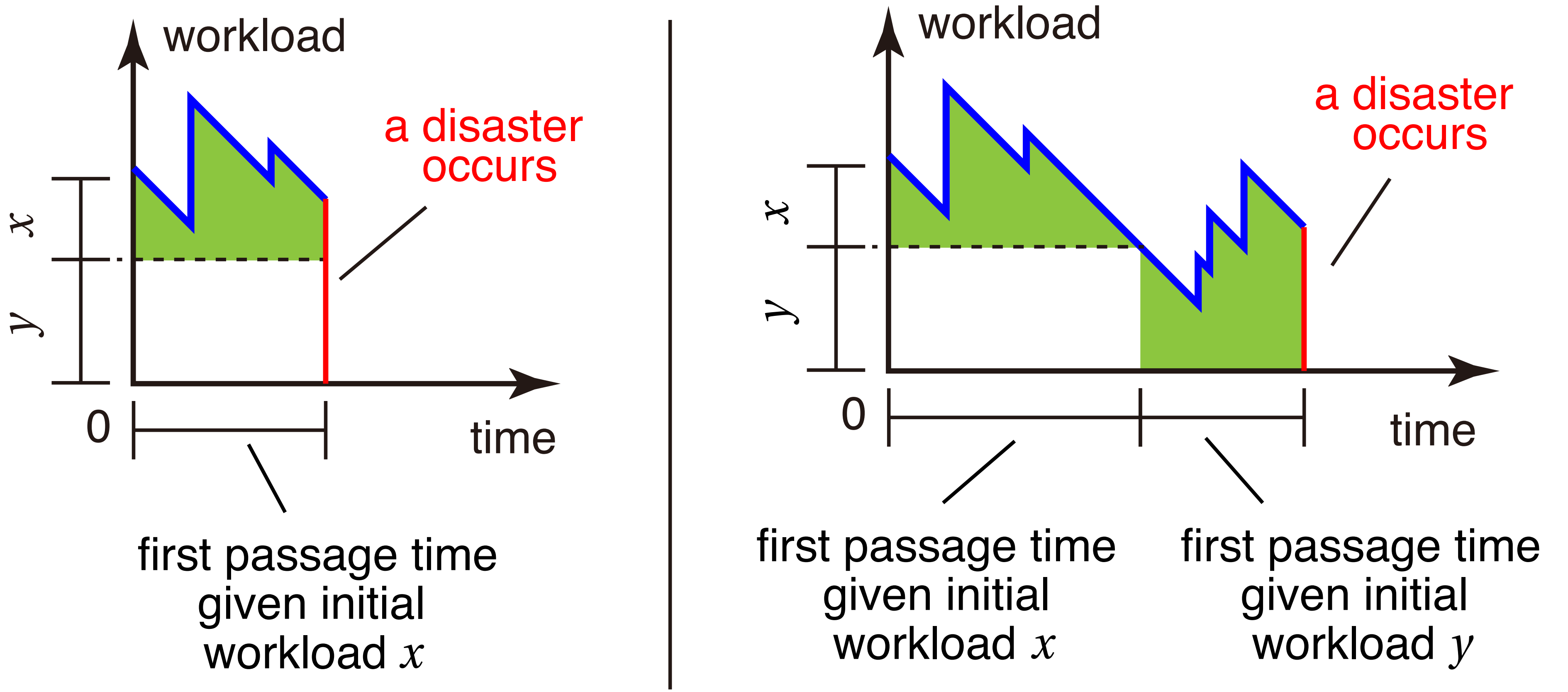

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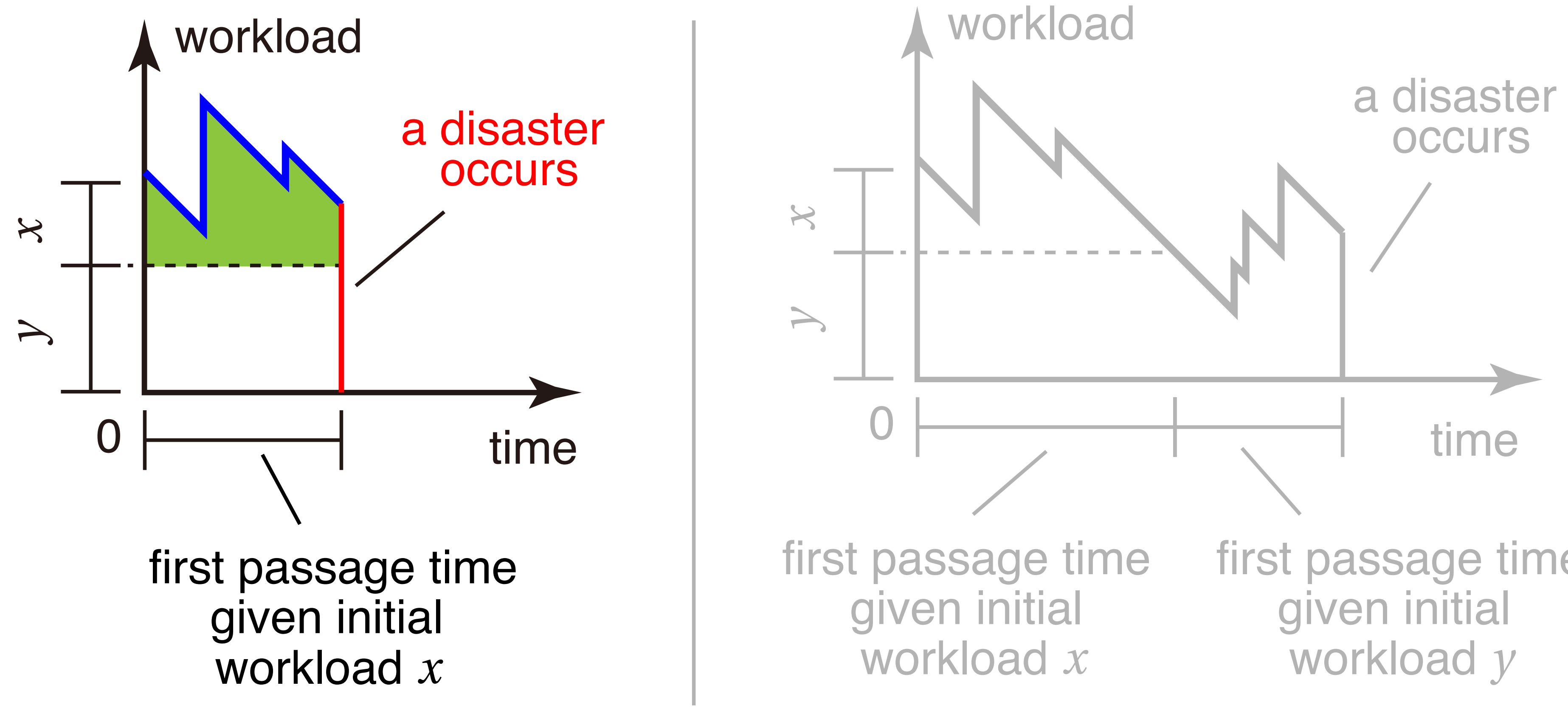

- A disaster occurs in the first passage time
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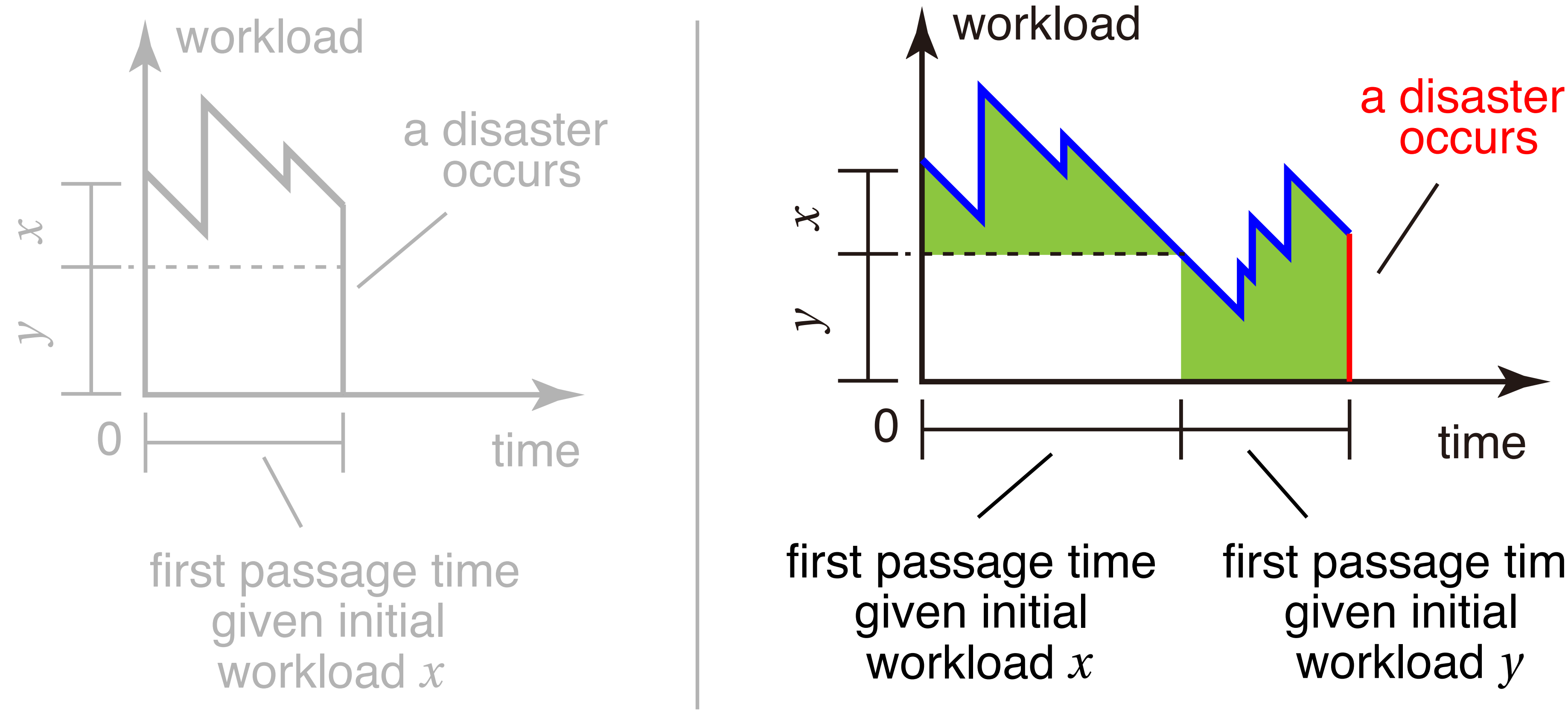

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Formulas for $\boldsymbol{P}_{\mathrm{N}}^{*}(s \mid x)$ and $\boldsymbol{P}_{\mathrm{D}}^{*}(s \mid x)$

- The properties of the first passage time yields
  - $$\boldsymbol{P}_{\mathrm{N}}^{*}(s \mid x) = \exp\left(\boldsymbol{Q}_{\mathrm{N}}^{*}(s)x\right) \quad (3)$$
  - $$\boldsymbol{P}_{\mathrm{D}}^{*}(s \mid x) = \int_{0}^{x} \exp\left(\boldsymbol{Q}_{\mathrm{N}}^{*}(s)w\right)dw \cdot \boldsymbol{Q}_{\mathrm{D}}^{*}(s) \quad (5)$$

where $\boldsymbol{Q}_{\mathrm{N}}^{*}(s)$ and $\boldsymbol{Q}_{\mathrm{D}}^{*}(s)$ are defined as

- $$\boldsymbol{Q}_{\mathrm{N}}^{*}(s) = -s\boldsymbol{I} + \boldsymbol{C} + \int_{0}^{\infty} d\boldsymbol{D}(y)\boldsymbol{P}_{\mathrm{N}}^{*}(s \mid y) \quad (4)$$
- $$\boldsymbol{Q}_{\mathrm{D}}^{*}(s) = \boldsymbol{\Gamma} + \int_{0}^{\infty} d\boldsymbol{D}(y)\boldsymbol{P}_{\mathrm{D}}^{*}(s \mid y) \quad (6)$$

State transition in the first passage time

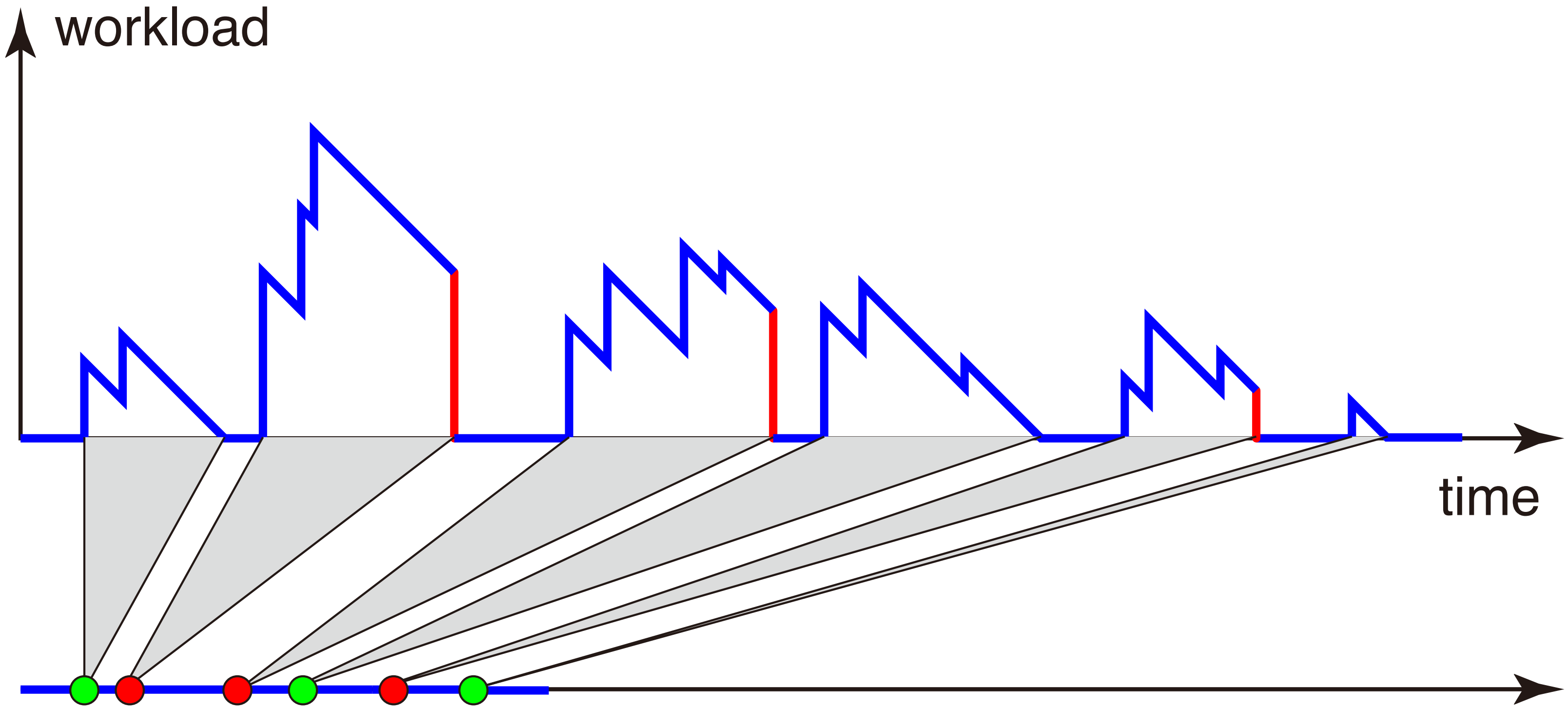
- $\boldsymbol{P}^*_{\mathrm{N}}(0 \mid x) = \lim_{s\to 0+} \boldsymbol{P}^*_{\mathrm{N}}(s \mid x), \quad \boldsymbol{P}^*_{\mathrm{D}}(0 \mid x) = \lim_{s\to 0+} \boldsymbol{P}^*_{\mathrm{D}}(s \mid x)$
  - $[\boldsymbol{P}^*_{\mathrm{N}}(0 \mid x)]_{i,j} = \Pr(S_{T_{\mathrm{E}}} = j,$ no disasters occur $\mid U_0 = x,\ S_0 = i)$
  - $[\boldsymbol{P}^*_{\mathrm{D}}(0 \mid x)]_{i,j} = \Pr(S_{T_{\mathrm{E}}} = j,$ a disaster occurs $\mid U_0 = x,\ S_0 = i)$
- $\boldsymbol{P}^*_{\mathrm{N}}(0 \mid x)$ and $\boldsymbol{P}^*_{\mathrm{D}}(0 \mid x)$ are given by
  - $\boldsymbol{P}^*_{\mathrm{N}}(0 \mid x) = \exp(\boldsymbol{Q}_{\mathrm{N}} x)$
  - $\boldsymbol{P}^*_{\mathrm{D}}(0 \mid x) = \int_0^x \exp(\boldsymbol{Q}_{\mathrm{N}} w)\, dw \cdot \boldsymbol{Q}_{\mathrm{D}}$

$$\boldsymbol{Q}_{\mathrm{N}} = \lim_{s\to 0+} \boldsymbol{Q}^*_{\mathrm{N}}(s) = \boldsymbol{C} + \int_0^\infty d\boldsymbol{D}(y) \boldsymbol{P}^*_{\mathrm{N}}(0 \mid y) \tag{9}$$

$$\boldsymbol{Q}_{\mathrm{D}} = \lim_{s\to 0+} \boldsymbol{Q}^*_{\mathrm{D}}(s) = \boldsymbol{\Gamma} + \int_0^\infty d\boldsymbol{D}(y) \boldsymbol{P}^*_{\mathrm{D}}(0 \mid y) \tag{10}$$

Probabilistic Interpretation of $\boldsymbol{Q}_{\mathrm{N}}$ and $\boldsymbol{Q}_{\mathrm{D}}$

- Remove all busy periods from the time axis
  - $\boldsymbol{Q}_{\mathrm{N}} + \boldsymbol{Q}_{\mathrm{D}}$ represents the infinitesimal generator of the resultant censored underlying Markov chain



Probabilistic Interpretation of $\boldsymbol{Q}_{\mathrm{N}}$ and $\boldsymbol{Q}_{\mathrm{D}}$

$\boldsymbol{Q}_{\mathrm{N}} + \boldsymbol{Q}_{\mathrm{D}}$: The infinitesimal generator of the censored underlying Markov chain

- $$\boldsymbol{Q}_{\mathrm{N}} = \boldsymbol{C} + \int_0^\infty d\boldsymbol{D}(y)\boldsymbol{P}_{\mathrm{N}}^*(0 \mid y) \qquad (9)$$
  - $\boldsymbol{Q}_{\mathrm{N}}$ represents the deficit infinitesimal generator when
    - neither arrivals nor disasters occur
    - busy periods without disasters are removed
- $$\boldsymbol{Q}_{\mathrm{D}} = \boldsymbol{\Gamma} + \int_0^\infty d\boldsymbol{D}(y)\boldsymbol{P}_{\mathrm{D}}^*(0 \mid y) \qquad (10)$$
  - $\boldsymbol{Q}_{\mathrm{D}}$ represents the transition rate matrix when
    - disasters occur in the idle state
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Computation of $\boldsymbol{Q}_{\mathrm{N}}$ and $\boldsymbol{Q}_{\mathrm{D}}$

- We define $\boldsymbol{Q}_{\mathrm{N}}^{(n)}$ ($n = 0, 1, \ldots$) by the following recursion

$$\boldsymbol{Q}_{\mathrm{N}}^{(0)} = \boldsymbol{C}, \quad \boldsymbol{Q}_{\mathrm{N}}^{(n)} = \boldsymbol{C} + \int_0^{\infty} d\boldsymbol{D}(y) \exp\big(\boldsymbol{Q}_{\mathrm{N}}^{(n-1)} y\big) \tag{19}$$

  - $\boldsymbol{Q}_{\mathrm{N}}^{(n)}$ is an elementwise increasing sequence of matrices

- $\boldsymbol{Q}_{\mathrm{N}}$ is given by $\boldsymbol{Q}_{\mathrm{N}} = \lim_{n\to\infty} \boldsymbol{Q}_{\mathrm{N}}^{(n)}$

- $\boldsymbol{Q}_{\mathrm{D}}$ is given by $\boldsymbol{Q}_{\mathrm{D}} = (-\boldsymbol{Q}_{\mathrm{N}})\big[-(\boldsymbol{C} + \boldsymbol{D})\big]^{-1} \boldsymbol{\Gamma}$ (13)

  - $\boldsymbol{C} + \boldsymbol{D}$: The deficit infinitesimal generator of the underlying Markov chain when no disasters occur

    - ➡ It is nonsingular

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Mean First Passage Time

- $\boldsymbol{f}(x)$: An $M \times 1$ vector whose ith element is given by
  - $[\boldsymbol{f}(x)]_i = \mathrm{E}[\text{The first passage time} \mid U_0 = x, S_0 = i]$
- $$\boldsymbol{f}(x) = (-1) \cdot \lim_{s \to 0+} \frac{\partial}{\partial s} \left[\boldsymbol{P}_{\mathrm{N}}^{*}(s \mid x) + \boldsymbol{P}_{\mathrm{D}}^{*}(s \mid x) \right] \boldsymbol{e} \qquad (20)$$
  - $\boldsymbol{e}$: An $M \times 1$ vector whose elements are all equal to one
- $\boldsymbol{f}(x)$ is given in terms of $\boldsymbol{Q}_{\mathrm{N}}$

$$\boldsymbol{f}(x) = \left[\boldsymbol{I} - \exp\left(\boldsymbol{Q}_{\mathrm{N}} x \right) \right] \left[-(\boldsymbol{C} + \boldsymbol{D}) \right]^{-1} \boldsymbol{e} \qquad (21)$$

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Idle Probability

- $\boldsymbol{\kappa}$: The conditional steady state probability vector, given that the system is idle
  - $[\boldsymbol{\kappa}]_j = \lim_{t\to\infty} \Pr(S_t = j \mid U_t = 0)$
- ν : The steady state probability that the system is busy
  - $\nu = \lim_{t\to\infty} \Pr(U_t > 0)$
- $\boldsymbol{\kappa}$ and ν are given in terms of $\boldsymbol{Q}_{\mathrm{N}}$ and $\boldsymbol{Q}_{\mathrm{D}}$
  - $\boldsymbol{\kappa}$ is determined uniquely by

$$\boldsymbol{\kappa}(\boldsymbol{Q}_{\mathrm{N}} + \boldsymbol{Q}_{\mathrm{D}}) = \boldsymbol{0}, \quad \boldsymbol{\kappa}\boldsymbol{e} = 1 \tag{17}$$

  - ν is given by

$$\nu = 1 - \frac{1}{\boldsymbol{\kappa}(-\boldsymbol{Q}_{\mathrm{N}})[-(\boldsymbol{C}+\boldsymbol{D})]^{-1}\boldsymbol{e}} \tag{32}$$

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Two Different Representations of the LST of the Stationary Distribution of Work in System

Work in System

U_t : The amount of work in system at time t

S_t : The state of the underlying Markov chain at time t

- $\boldsymbol{u}_t(x)$: A $1 \times M$ vector whose jth element is given by
  - $\left[\boldsymbol{u}_t(x)\right]_j = \Pr(U_t \le x, S_t = j)$
- We define $1 \times M$ vectors $\boldsymbol{u}(x)$ and $\boldsymbol{u}^*(s)$ as
  - $\boldsymbol{u}(x) = \lim_{t \to \infty} \boldsymbol{u}_t(x)$
  - $\boldsymbol{u}^*(s) = \int_0^\infty \exp(-sx) d\boldsymbol{u}(x)$
- We derive two different representations of $\boldsymbol{u}^*(s)$

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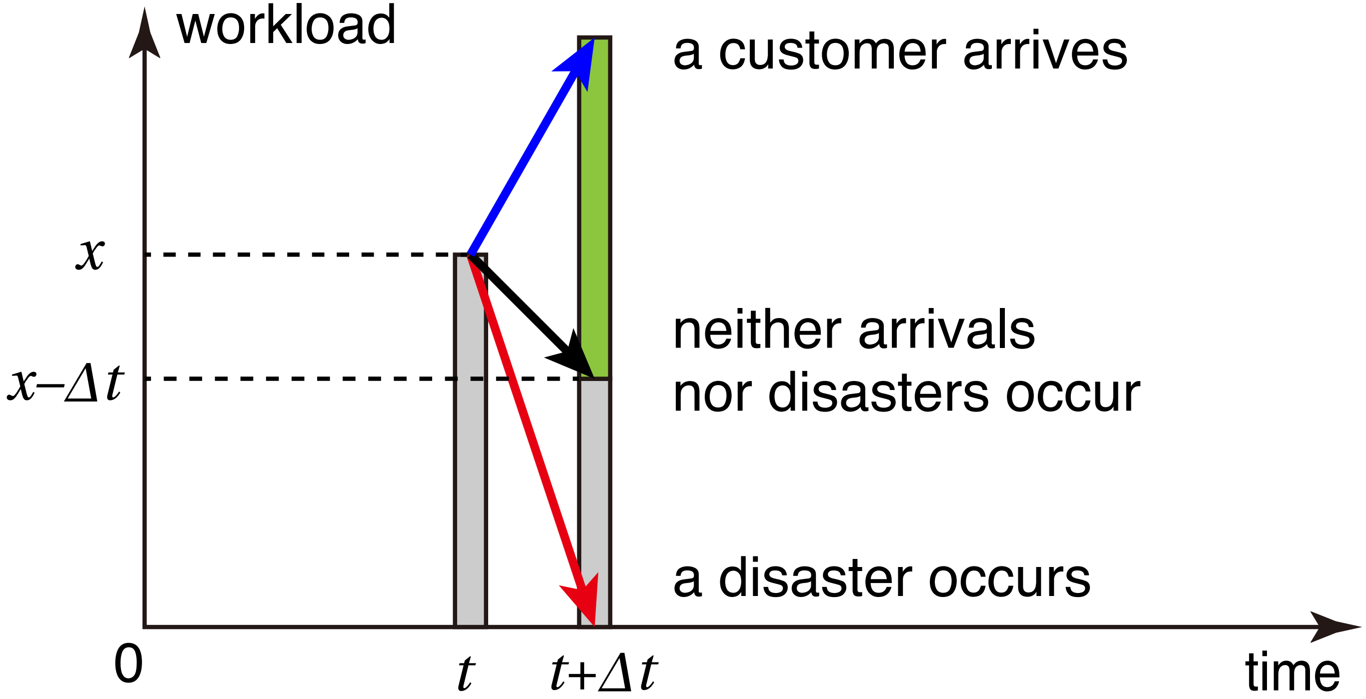
Transition from time t to $t+\Delta t$



- $$\boldsymbol{u}_{t+\Delta t}(x) = \boldsymbol{u}_t(x+\Delta t)\boldsymbol{C}\Delta t + \int_0^x \boldsymbol{u}_t(x-y+\Delta t)\,d\boldsymbol{D}(y)\Delta t + \boldsymbol{u}_t(\infty)\boldsymbol{\Gamma}\Delta t + \boldsymbol{o}(\Delta t)$$

LST of the Workload Distribution $\boldsymbol{u}^*(s)$

- $$\boldsymbol{u}_{t+\Delta t}(x) = \boldsymbol{u}_t(x+\Delta t)\boldsymbol{C}\Delta t + \int_0^x \boldsymbol{u}_t(x-y+\Delta t)\,d\boldsymbol{D}(y)\Delta t + \boldsymbol{u}_t(\infty)\boldsymbol{\Gamma}\Delta t + \boldsymbol{o}(\Delta t)$$

$$\Rightarrow \frac{\partial}{\partial t}\left[\boldsymbol{u}_t(x)\right] = \frac{\partial}{\partial x}\left[\boldsymbol{u}_t(x)\right] + \boldsymbol{u}_t(x)\boldsymbol{C} + \int_0^\infty \boldsymbol{u}_t(x-y)\,d\boldsymbol{D}(y) + \boldsymbol{u}_t(\infty)\boldsymbol{\Gamma}$$

- Take the limit $t \to \infty$
  - $$0 = \frac{d}{dx}\left[\boldsymbol{u}(x)\right] + \boldsymbol{u}(x)\boldsymbol{C} + \int_0^\infty \boldsymbol{u}(x-y)\,d\boldsymbol{D}(y) + \boldsymbol{\pi}\boldsymbol{\Gamma}$$

- Take the LST with respect to x

$$\boldsymbol{u}^*(s)\left[s\boldsymbol{I} + \boldsymbol{C} + \boldsymbol{D}^*(s)\right] = s(1-\nu)\boldsymbol{\kappa} - \boldsymbol{\pi}\boldsymbol{\Gamma} \qquad (33)$$

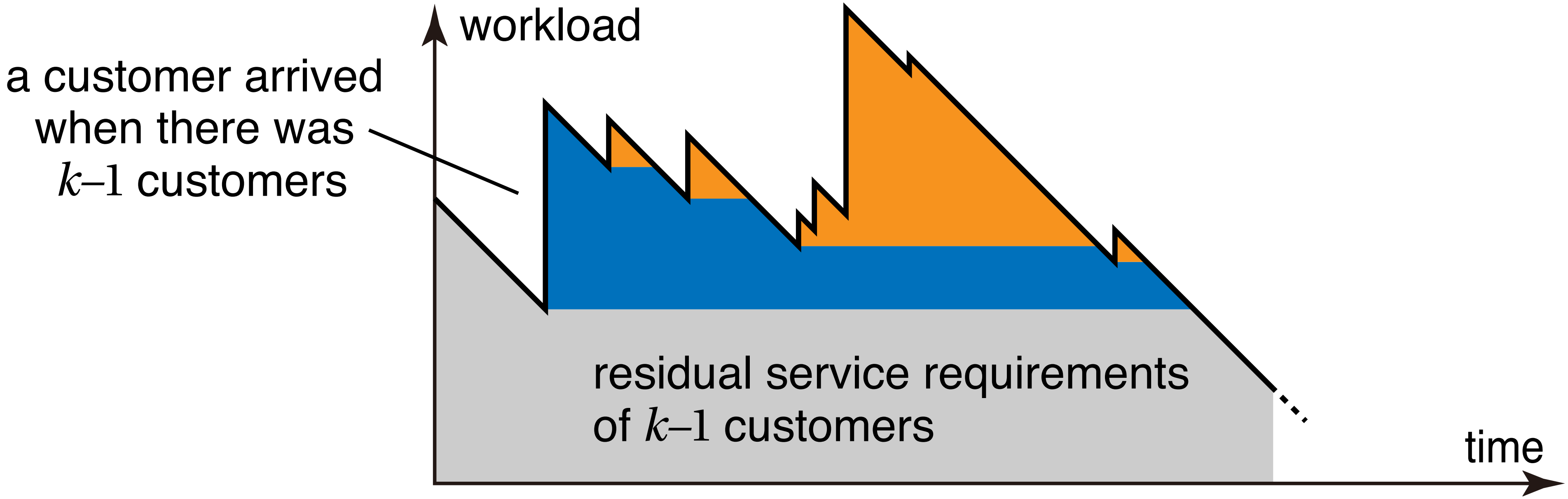
Preemptive-Resume LIFO

- Assume that customers are served on a preemptive-resume LIFO basis
  - This service discipline is work conserving
- $\boldsymbol{u}^*(s,k)$: The LST of the workload distribution when k customers are present in the system
  - $\boldsymbol{u}^*(s,0) = (1-\nu)\boldsymbol{\kappa}$
  - $\boldsymbol{u}^*(s) = \sum_{k=0}^{\infty} \boldsymbol{u}^*(s,k)$

Alternavite Representation of $u^*(s)$

When there is k customers in the system

- Workload in system is equal to the sum of the remaining service requirements of
  - $k-1$ waiting cusotmers
  - the customer being served



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- $$\boldsymbol{u}^*(s,k) = \boldsymbol{u}^*(s,k-1)\boldsymbol{R}^*(s), \quad k = 1,2,\ldots \tag{41}$$
  - $$\boldsymbol{R}^*(s) = \int_0^\infty \exp(-sx)\,dx \int_x^\infty d\boldsymbol{D}(y)\exp\big(\boldsymbol{Q}_{\mathrm{N}}(y-x)\big) \tag{38}$$
- $$\boldsymbol{u}^*(s) = \sum_{k=0}^{\infty} \boldsymbol{u}^*(s,k) = \boxed{(1-\nu)\boldsymbol{\kappa}\left[\boldsymbol{I}-\boldsymbol{R}^*(s)\right]^{-1}} \tag{45}$$
  - It is shown that $\boldsymbol{I}-\boldsymbol{R}^*(s)$ ($\mathrm{Re}(s) > 0$) is nonsingular

Conclusion

- We considered the workload distribution
  in a MAP/G/1 queue with disasters
  - We derived two different formulas

$$\boldsymbol{u}^*(s)\left[s\boldsymbol{I}+\boldsymbol{C}+\boldsymbol{D}^*(s)\right] = (1-\nu)\boldsymbol{\kappa}-\boldsymbol{\pi}\boldsymbol{\Gamma} \tag{33}$$

$$\boldsymbol{u}^*(s) = (1-\nu)\boldsymbol{\kappa}\left[\boldsymbol{I}-\boldsymbol{R}^*(s)\right]^{-1} \tag{45}$$

  - We showed that these formulas are equivalent
    in a sense that one can be derived from another
- We have already finished an additional analysis

  on the queue length, the waiting time, and the sojourn time
  in a FIFO MAP/G/1 queue with disasters
- We are going to submit a paper with these results to JIMO