The Stationary Distribution of the Age of Information in FCFS Single-Server Queues

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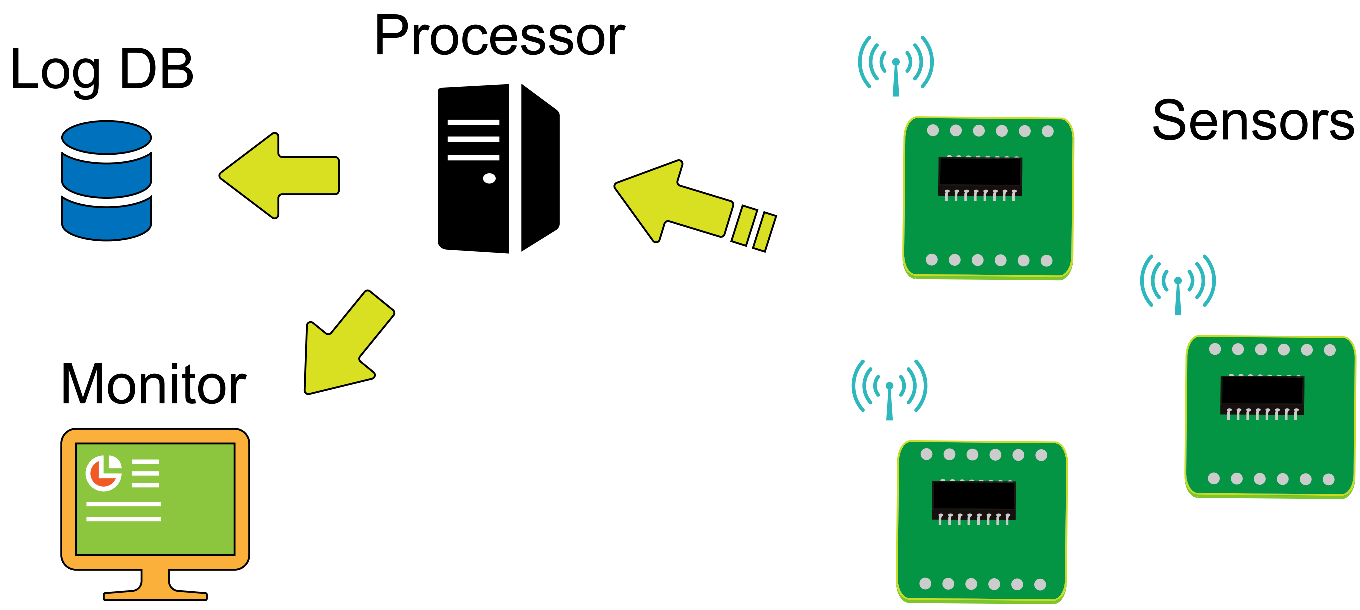
Information Update Systems

- The state of an information source is monitored over time



- Abstraction of various situations where the freshness of data is of interest
  - Satellite imagery, tracking trends in SNS, and so on

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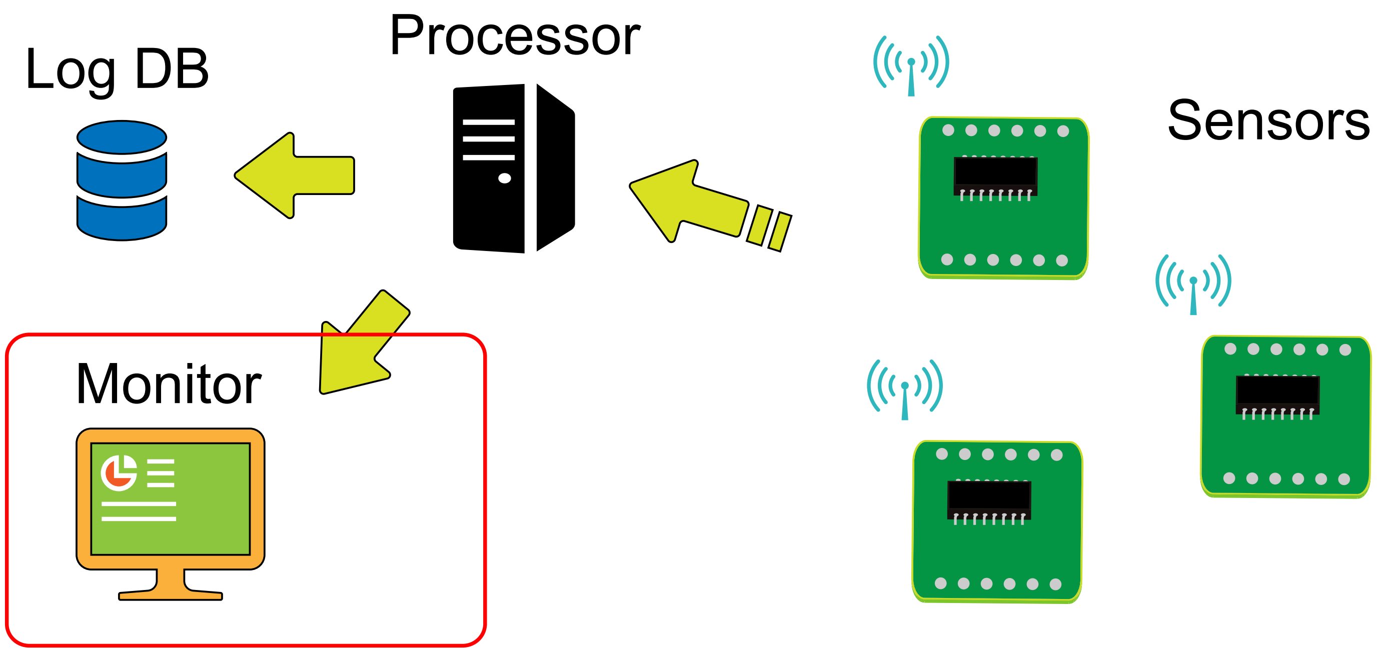

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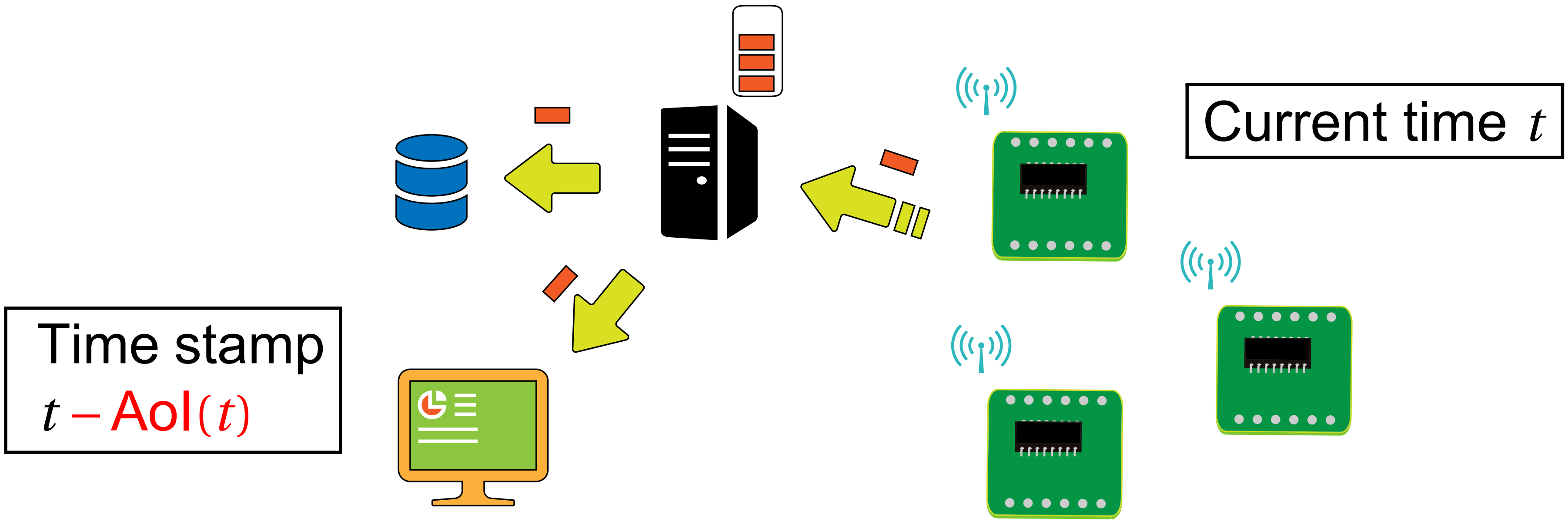
  - Satellite imagery, tracking trends in SNS, and so on

Age of Information (AoI)

AoI $\triangleq$ Current time – Time-stamp of the displayed information

- This system is usually modeled as queueing systems
  - Information packets are regarded as customers arriving to a queueing system

➨ The AoI is formulated as a continuous-time stochastic process



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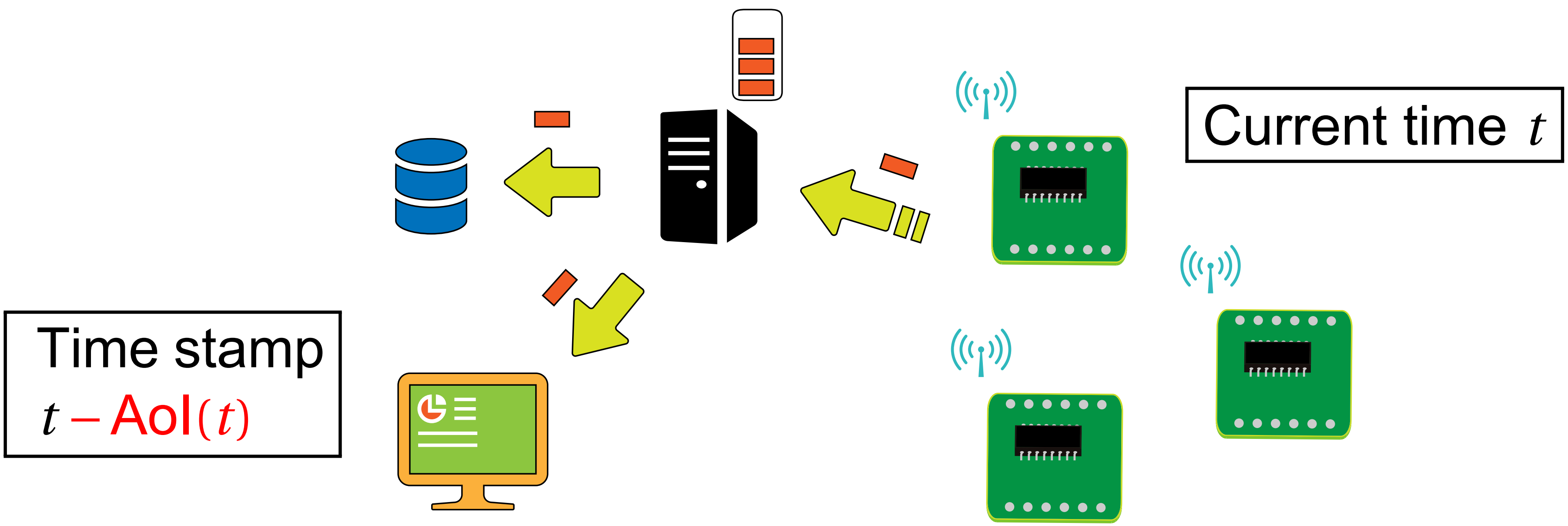


Related Works (1)

The AoI in first-come first-served (FCFS) queues

- M/M/1, M/D/1, and D/M/1 queues [1]
  - Standard single-processor models
- M/M/2 and M/M/∞ queues [2]
  - Modeling out-of-order packet arrivals via a dynamic network
- M/M/1/1 and M/M/1/2 queues [3]
  - The waiting buffer has finite capacity (zero or one)

[1] S. Kaul et al., in Proc. of IEEE INFOCOM 2012, 2731–2735, Mar. 2012.
[2] C. Kam. et al., IEEE Trans. Inf. Theory, 62, 1360–1374, Mar. 2016.
[3] M. Costa et al., IEEE Trans. Inf. Theory, 62, 1897–1910, Apr. 2016.

Related works (2)

The AoI in last-come first-served (LCFS) queues

- Preemptive M/M/1/1 and non-preemptive M/M/1/2 queues [4]

  with and without preemption of service on arrival

- Preemptive M/Gamma/1/1 queue
  and non-preemptive M/Er/1/2 queue [5]

  - Results in [4] are generalized
    w.r.t. the processing time distribution

[4] S. Kaul et al., in Proc. of CISS 2012, Mar. 2012.

[5] E. Najm and R. Nasser, in Proc. of IEEE ISIT 2016, 2574–2578, Jul. 2016.

Related Works (3)

Systems with additional features, and the optimal control of the AoI

[6] B. T. Bacinoglu et al. in Proc. of 2015 ITA Workshop, 25–31, Feb. 2015.

[7] N. Pappas et al., in Proc. of IEEE ICC 2015, 5935–5940, Jun. 2015.

[8] R. D. Yates, in Proc. of IEEE ISIT 2015, 3008–3012, Jun. 2015.

[9] Y. Sun et al. in Proc. of IEEE INFOCOM 2016, Apr. 2016.

[10] Q. He et al., in Proc. of IEEE ICC 2016, May 2016.

[11] Q. He et al., in Proc. of WiOpt 2016, May 2016.

[12] A. M. Bedewy et al., in Proc. of IEEE ISIT 2016, 2569–2573, Jul. 2016.

[13] K. Chen and L. Huang., in Proc. of IEEE ISIT 2016, 2579–2583, Jul. 2016.

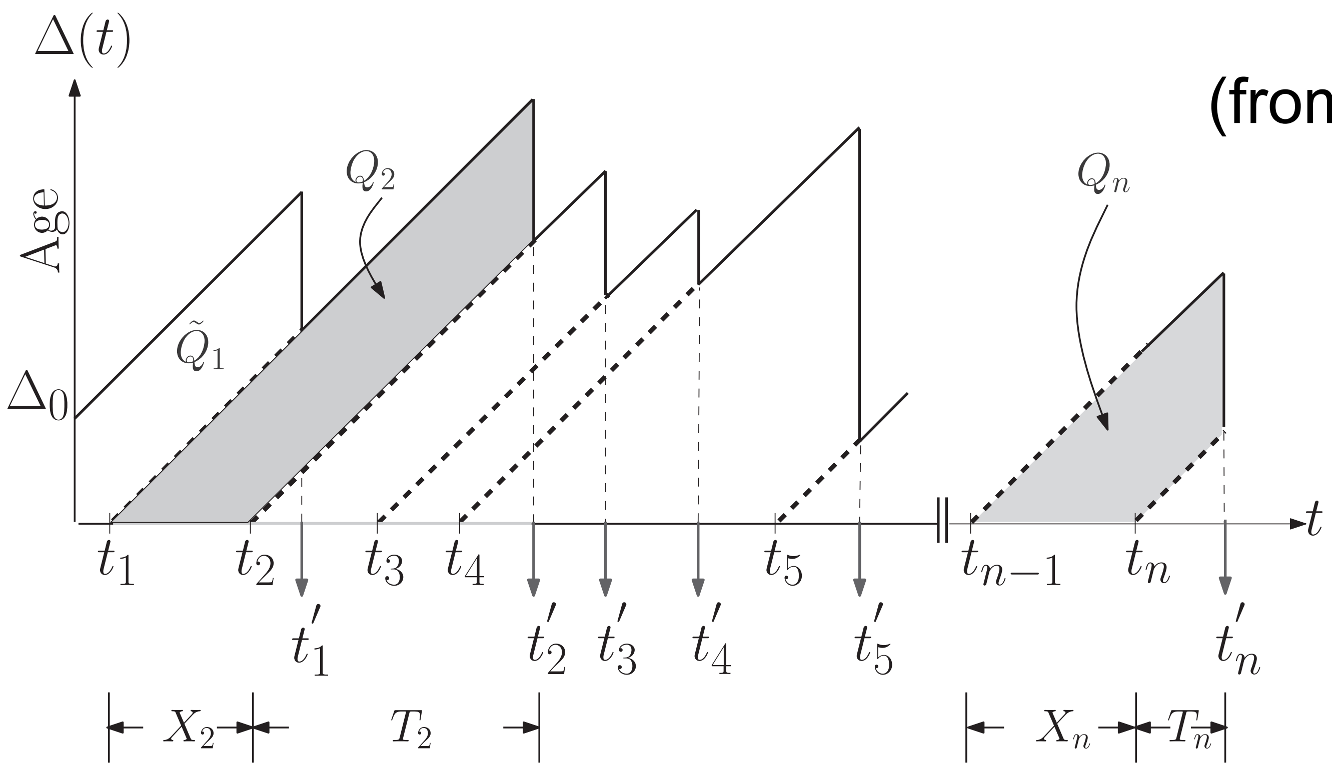
[14] C. Kam et al. in Proc. of IEEE ISIT 2016, 2564–2568, Jul. 2016.

[15] C. Kam et al., in Proc. of IEEE Milcom 2016, 301–306, Nov. 2016.

Motivation (1)

- In most previous works, only the mean AoI is considered

$$\text{The mean AoI} = \lim_{T\to\infty} \frac{\text{The area under the graph of the AoI process}}{\text{Observation duration } T}$$



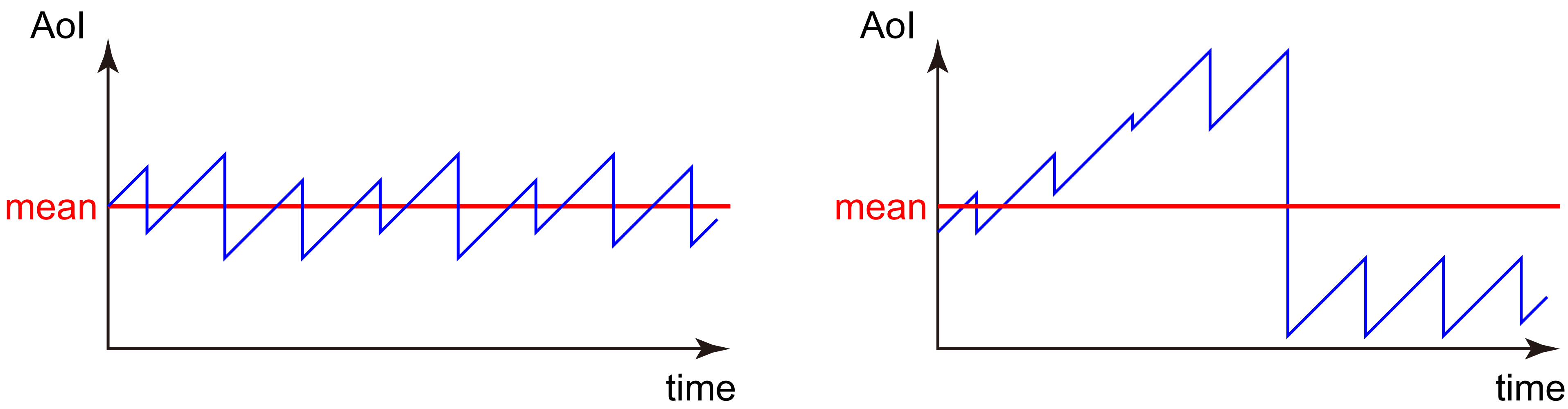
(from Fig. 2 in [1])

[1] S. Kaul et al., in Proc. of IEEE INFOCOM 2012, 2731–2735, Mar. 2012.

Motivation (2)

- Although the mean AoI is a primary performance measure, it alone is not sufficient to characterize the AoI

E.g.) The deviation of the AoI from its mean value

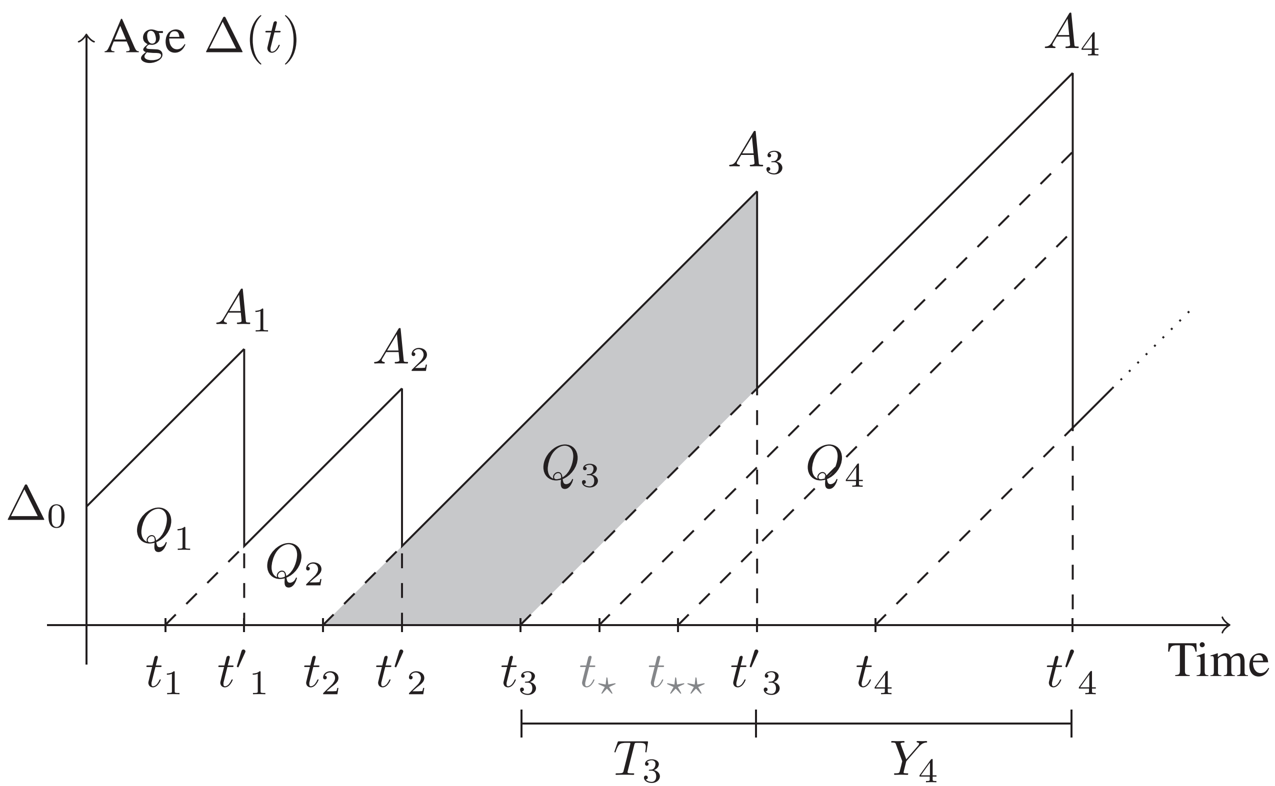


- We are thus interested in the probability distribution of the AoI

Peak AoI

- The probability distribution of the peak AoI is analyzed in [3]
  - The AoI immediately before information updates

(from Fig. 3 in [3])



[3] E. Najm and R. Nasser, in Proc. of IEEE ISIT 2016, 2574–2578, Jul. 2016.

Outline of This Talk

- We consider the probability distribution $A(x)$ of the AoI

  $A(x)$: The long-run fraction of time that the AoI $\leq x$

- We derive an invariant relation among the distributions of the AoI, the peak AoI, and the system delay,

  which holds for a wide class of information update systems

Based on this relation,

- We analyze the AoI in the FCFS GI/GI/1 queue
  - The distribution of the AoI is given in terms of the system delay distribution

➡ We specialize this result to the M/GI/1 and GI/M/1 queues

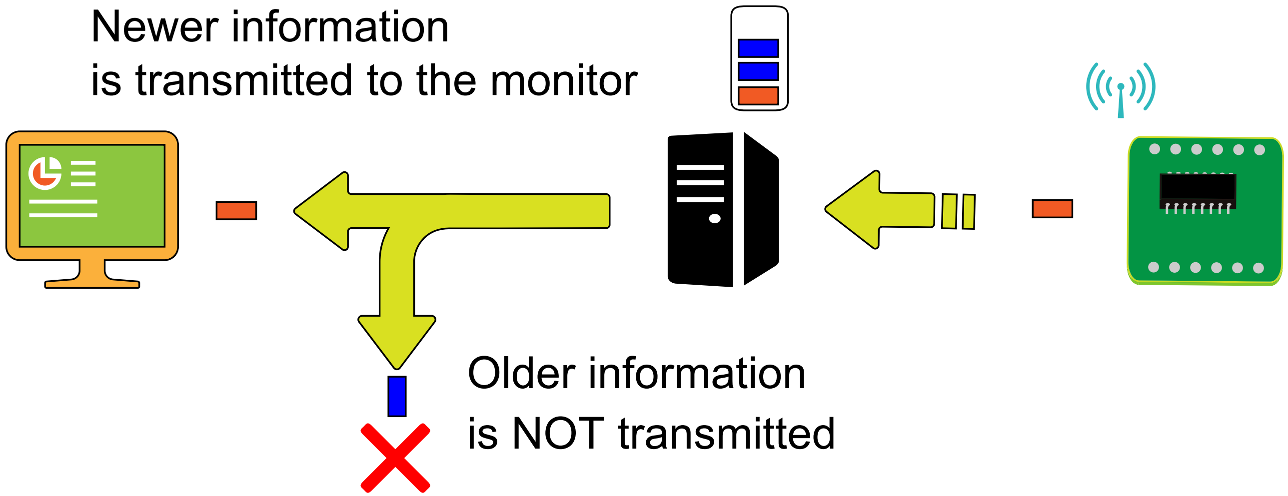
Invariant Relation for the AoI Distribution

Types of Information Packets

- In general, information packets are categorized into two types
  - Informative packets, which contain newer information
  - Non-informative packets, which contain older information

E.g.) An FCFS system ➨ All packets are informative

An LCFS system ➨ Non-informative packets exist



- If we observe only the stream of informative packets, we have a FIFO (First-In-First-Out) queueing system

Formulation of General Sample-Paths

We consider a FIFO queue of informative packets

- A sample-path of a general FIFO queue is characterized by

  α_n $(n = 0, 1 \ldots)$: The arrival time of the nth packet

  β_n $(n = 0, 1 \ldots)$: The departure time of the nth packet

  α_n and β_n are deterministic sequences (not random variables)

- We assume the followings

  (i) $\alpha_n \leq \alpha_{n+1}$ (Packets are numbered in order of arrival)

  (ii) $\alpha_n \leq \beta_n$ (A packet cannot depart before its arrival)

  (iii) $\beta_n \leq \beta_{n+1}$ (Packets depart in a FIFO manner)

  (iv) $\alpha_0 \leq \beta_0 = 0 < \alpha_1$ (The system becomes empty at time 0)

AoI and Peak AoI

- M_t: The index of the last departed packet

$$M_t = \sum_{n=1}^{\infty} \mathbb{1}\{\beta_n \le t\}$$

$$\mathbb{1}\{X\} \triangleq \begin{cases} 1, & X \text{ is true} \\ 0, & X \text{ is false} \end{cases}$$

- A_t: The AoI at time t

$$A_t = t - \alpha_{M_t} \qquad \text{(Current Time – Time-Stamp)}$$

- $A_{\text{peak},n}$: The nth peak AoI

$$A_{\text{peak},n} = \lim_{\Delta t \to 0+} A_{\beta_n - \Delta t} \qquad \text{(just before the } n\text{th departure)}$$

- We assume that the arrival rate λ is positive and finite

$$\lambda = \lim_{T \to \infty} \frac{1}{T} \sum_{n=1}^{\infty} \mathbb{1}\{\alpha_n \le T\} \in (0, \infty)$$

An Invariant Relation

A_t: The AoI at time t $A_{\text{peak},n}$: The nth peak AoI

D_n: The system delay of the nth packet $(= \beta_n - \alpha_n)$

- $A^\sharp(x) \triangleq \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbb{1}\{A_t \le x\}\,\mathrm{d}t$ (The fraction of time with $A_t \le x$)
- $A^\sharp_{\text{peak}}(x) \triangleq \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mathbb{1}\{A_{\text{peak},n} \le x\}$ (The relative number of peak AoIs with $A_{\text{peak},n} \le x$)
- $D^\sharp(x) \triangleq \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mathbb{1}\{D_n \le x\}$ (The relative number of packets with $D_n \le x$)

Under some regularity conditions (convergence and stability),

$$A^\sharp(x) = \lambda \int_0^x \left(D^\sharp(y) - A^\sharp_{\text{peak}}(y)\right)\mathrm{d}y$$

Applications to FCFS Single-Server Queues

GI/GI/1 Queue (1)

- Inter-arrival times are assumed to be i.i.d. with
  - general probability distribution function $G(x)$
  - mean $\mathrm{E}[G]$ ($\lambda = 1/\mathrm{E}[G]$ follows)
- Processing times are assumed to be i.i.d. with
  - general probability distribution function $H(x)$
  - mean $\mathrm{E}[H]$
- The traffic intensity $\rho \triangleq \lambda \mathrm{E}[H]$
  - We assume $\rho < 1$ so that the system is stable

GI/GI/1 Queue (2)

In addition, we assume that the system is stationary and ergodic

- Probability distributions of system-states are time-invariant
  - They are called stationary distributions, which the queueing theory usually deals with

- Stationary distributions = Probability distributions on a sample-path

➡ $A(x) = A^{\sharp}(x)$, $A_{\mathrm{peak}}(x) = A^{\sharp}_{\mathrm{peak}}(x)$, and $D(x) = D^{\sharp}(x)$

$A(x)$: The stationary AoI distribution

$A_{\mathrm{peak}}(x)$: The stationary peak AoI distribution

$D(x)$: The stationary system delay distribution

The Invariant Relation (GI/GI/1)

For any non-negative random variable F, we define

$$F(x) = \Pr(F \le x), \quad f(x) = \frac{dF(x)}{dx}, \quad f^*(s) = \int_0^\infty e^{-sx} \mathrm{d}F(x) \quad \text{(LST of } F\text{)}$$

- The stationary AoI distribution is characterized by

$$a(x) = \frac{D(x) - A_{\mathrm{peak}}(x)}{\mathrm{E}[G]}, \quad \text{or equivalently,} \quad a^*(s) = \frac{d^*(s) - a^*_{\mathrm{peak}}(s)}{s\mathrm{E}[G]}$$

- In addition, we can show that

$$a^*_{\mathrm{peak}}(s) = \left[\int_0^\infty e^{-sx} G(x) \mathrm{d}D(x) + \int_0^\infty e^{-sx} D(x) \mathrm{d}G(x)\right] h^*(s)$$

➡ The AoI distribution in the GI/GI/1 queue is given in terms of the system delay distribution

Special Cases: M/GI/1 and GI/M/1 Queues

M/GI/1 Queue (1)

- Exponential inter-arrival time distribution $G(x) = 1 - e^{-\lambda x}$
- General processing time distribution $H(x)$ (LST $h^*(s)$)
  - M/M/1 and M/D/1 [1] are special cases of this model

In this model, the LST of the system delay D is given by

$$d^*(s) = \frac{(1-\rho)s}{s - \lambda + \lambda h^*(s)} \cdot h^*(s)$$

➡ We obtain the LST of the stationary AoI distribution

$$a^*(s) = \rho d^*(s) \cdot \frac{1 - h^*(s)}{\mathrm{E}[H]s} + d^*(s+\lambda) \cdot \frac{\lambda}{s+\lambda} \cdot h^*(s)$$

[1] S. Kaul et al., in Proc. of IEEE INFOCOM 2012, 2731–2735, Mar. 2012.

M/GI/1 Queue (2)

- The first two moments of the AoI distribution are given by

$$\mathrm{E}[A] = \mathrm{E}[D] + \frac{1-\rho}{\rho h^*(\lambda)} \cdot \mathrm{E}[H]$$

$$\mathrm{E}[A^2] = \mathrm{E}[D^2] + \frac{2(1-\rho)}{(\rho h^*(\lambda))^2} \left[1 + \rho h^*(\lambda) + \lambda h^{(1)}(\lambda)\right] (\mathrm{E}[H])^2$$

where

$$\mathrm{E}[D] = \frac{\lambda \mathrm{E}[H^2]}{2(1-\rho)} + \mathrm{E}[H]$$

$$\mathrm{E}[D^2] = \frac{\lambda \mathrm{E}[H^3]}{3(1-\rho)} + \frac{(\lambda \mathrm{E}[H^2])^2}{2(1-\rho)^2} + \frac{\mathrm{E}[H^2]}{1-\rho}$$

GI/M/1 Queue (1)

- General inter-arrival time distribution $G(x)$ (LST $g^*(s)$)
- Exponential processing time distribution $H(x) = 1 - e^{-\mu x}$
  - D/M/1 queue [1] is a special case of this model

In this model, the system delay distribution is exponential

$$D(x) = 1 - e^{-(1-\gamma)\mu x}, \qquad d^*(s) = \frac{(1-\gamma)\mu}{s + (1-\gamma)\mu}$$

where γ is a unique solution of $\gamma = g^*(\mu - \mu\gamma)$ and $\gamma \in (0, 1)$

➡ We obtain the LST of the stationary AoI distribution

$$a^*(s) = \left[\rho \cdot \frac{(1-\gamma)\mu}{s + (1-\gamma)\mu} \cdot \frac{g^*(s + (1-\gamma)\mu) - \gamma}{1-\gamma} + \frac{1 - g^*(s)}{s\mathrm{E}[G]}\right] \frac{\mu}{s + \mu}$$

[1] S. Kaul et al., in Proc. of IEEE INFOCOM 2012, 2731–2735, Mar. 2012.

GI/M/1 Queue (2)

- The first two moments of the AoI distribution are given by

$$\mathrm{E}[A] = \frac{\mathrm{E}[G^2]}{2\mathrm{E}[G]} + \frac{1}{\mu} - \frac{g^{(1)}((1-\gamma)\mu)}{(1-\gamma)\mu\mathrm{E}[G]}$$

$$\mathrm{E}[A^2] = \frac{\mathrm{E}[G^3]}{3\mathrm{E}[G]} + \rho\mathrm{E}[G^2] + \frac{2}{\mu^2} + \frac{\rho}{1-\gamma}\left[g^{(2)}((1-\gamma)\mu) - 2\left(\frac{1}{(1-\gamma)\mu} + \frac{1}{\mu}\right)g^{(1)}((1-\gamma)\mu)\right]$$

Conclusion

- We considered the probability distribution of the AoI
- Under a general setting, we proved an invariant relation

$$A^{\sharp}(x) = \lambda \int_0^x \left(D^{\sharp}(y) - A^{\sharp}_{\mathrm{peak}}(y) \right) \mathrm{d}y$$

  - The distribution of the AoI is given in terms of the distributions of the peak AoI and the system delay

- Based on this result, we analyzed the AoI in FCFS queues
  - GI/GI/1, M/GI/1, and GI/M/1 queues
- A full version of the paper with results on standard LCFS queues is in preparation