Analysis of the Age of Information with Packet Deadline and Infinite Buffer Capacity

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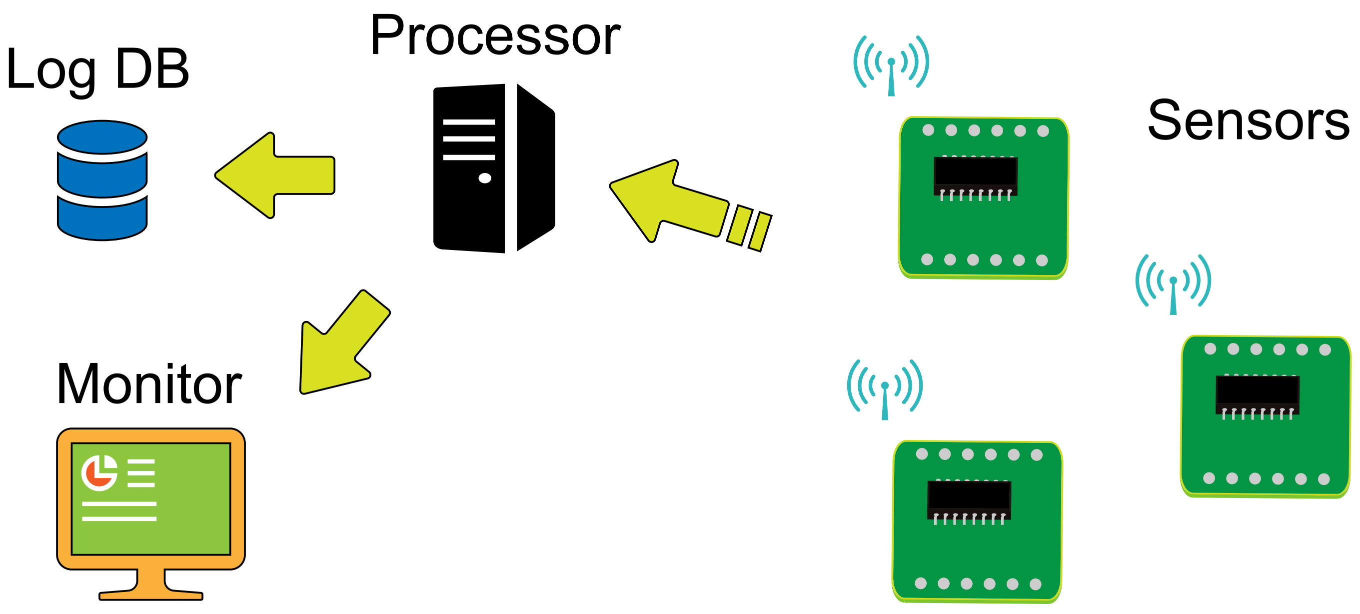
Information Update Systems

- The state of an information source is monitored over time



- Abstraction of various situations
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  - Satellite imagery, tracking trends in SNS, and so on

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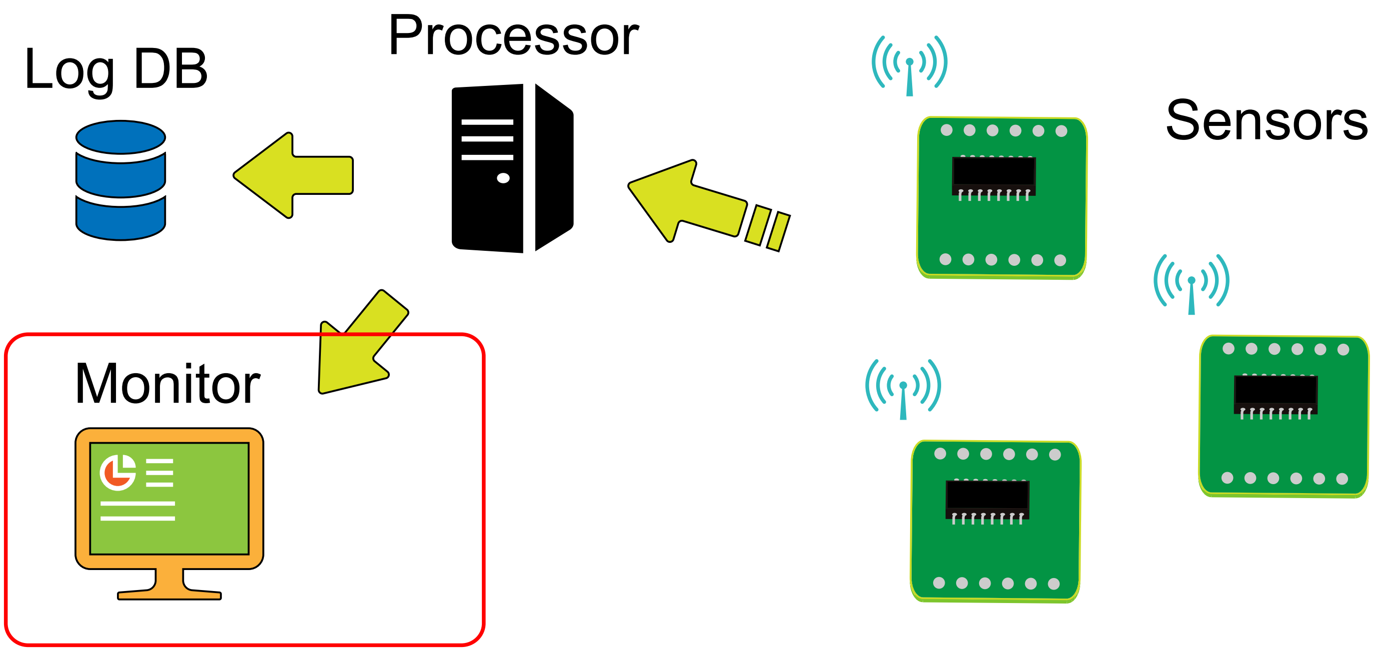

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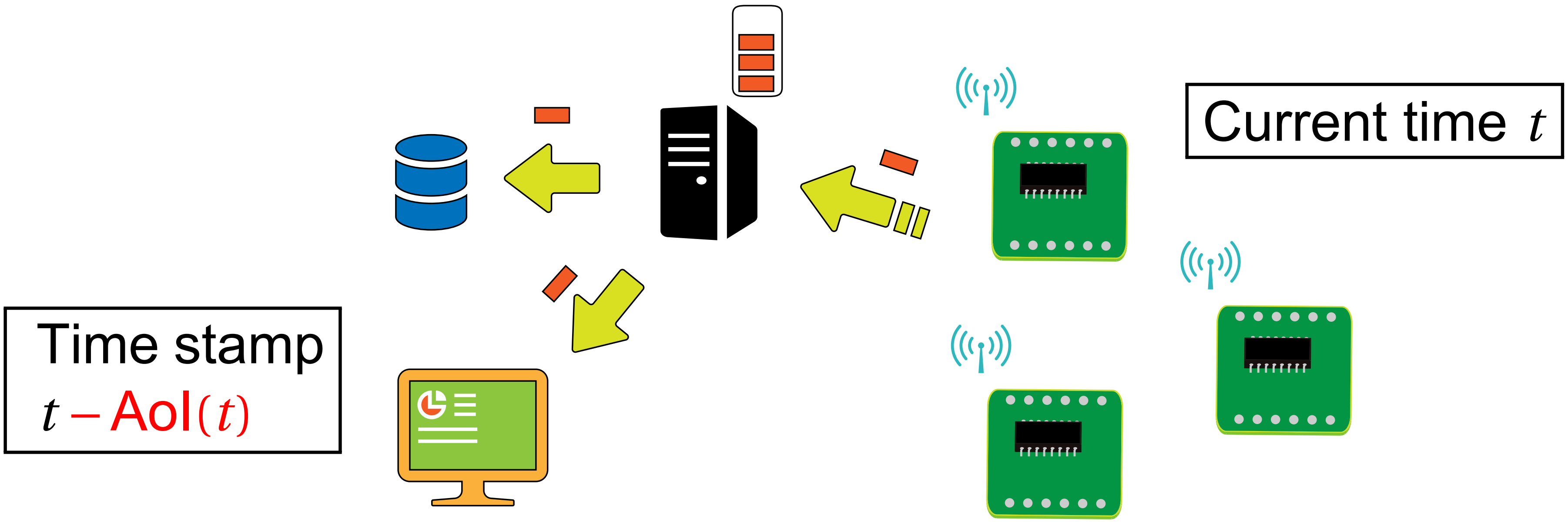
Age of Information (AoI)

AoI $\triangleq$ Current time – Time-stamp of the displayed information

- This system is usually modeled as queueing systems
  - Information packets are regarded as customers arriving to a queueing system

➨ The AoI is formulated as a continuous-time stochastic process



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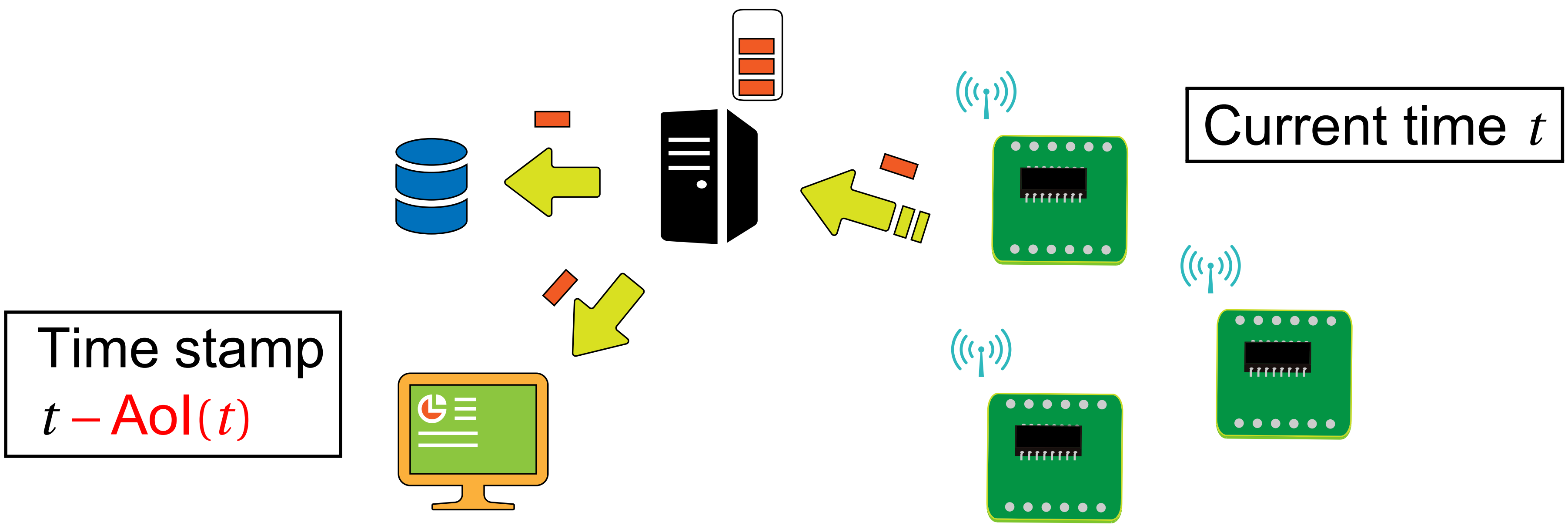


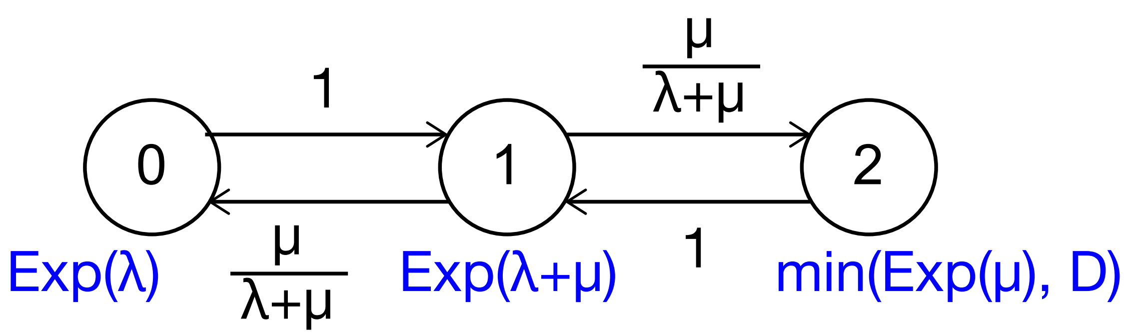
AoI with Packet Deadline (1)

- Imposing deadlines to packets reduces the AoI [1,2]
  - Waiting packet is dropped when its deadline expires
- In [1,2], explicit formulas for the mean AoI is derived for
  - M/M/1/2+M queue
    - Exponential services & Exponential deadlines
    - Easier to analyze because of the memoryless property
  - M/M/1/2+D queue
    - Exponential services & Constant deadlines
    - More difficult to analyze

[1] Kam et al., in Proc. of IEEE ISIT 2016 (2016).
[2] Kam el al., IEEE Trans. Inf. Theory (2018).

AoI with Packet Deadline (2)

- In the M/M/1/2+D queue, there is only one buffer space
  - ➨ It is sufficient to consider the remaining time to deadline for at most one packet
  - ➨ The number of packets in the system is formulated as a semi-Markov process with three-states [1,2]



- It is not straightforward to extend this approach to systems with buffer capacity larger than one
  - ➨ It is necessary to keep track of the deadlines of all waiting packets

Outline of This Talk

We consider the AoI with packet deadlines

- We analyze the probability distribution of the AoI,
  assuming Infinite buffer capacity
- We first analyze a general case (M/G/1+G queue), where
  - Service times follows a general non-negative distribution
  - Deadlines follows a general non-negative distribution
- Specializing the result, we obtain explicit formulas for
  - The density function $a(x)$ of the AoI in the M/M/1+G
    (Exponential services)
  - The mean AoI $\mathrm{E}[A]$ in the M/M/1+D
    (Expopential services & Constant deadlines)

AoI in the M/G/1+G Queue

Model

- Packets arrive according to a Poisson process
  - λ: Arrival rate of packets
- Service times of packets are i.i.d. with finite mean $\mathrm{E}[H]$
  - $H(x)$: CDF of service times
- Deadlines of packets are i.i.d.
  - $G(x)$: CDF of deadlines
- For simplicity, we assume $H(0) = G(0) = 0$

Informative and Non-Informative Packets

There are two types of packets in this model

- Informative packets
  - Packets which are eventually processed
- Non-informative packets
  - Packets which are lost due to deadline expiration

Application of a General Formula

$A(x)$: Stationary AoI distribution
(Long-run fraction of time that the AoI $\leq x$)

- $a^*(s)$: The Laplace-Stieltjes transform (LST) of $A(x)$

$$a^*(s) = \int_0^\infty e^{-sx} \mathrm{d}A(x), \qquad \mathrm{Re}(s) > 0$$

- The following relation holds under a fairly general setting [3,4]

$$a^*(s) = \lambda^\dagger \cdot \frac{d^*(s) - a^*_{\mathrm{peak}}(s)}{s}$$

$\lambda^\dagger$: Mean number of information updates per time unit

$d^*(s)$: LST of the system delay distribution of informative packets

$a^*_{\mathrm{peak}}(s)$: LST of the peak AoI distribution

[3] Inoue et al., in Proc of IEEE ISIT 2017 (2017).

[4] Inoue et al., arXiv preprint (2018).

Approach of the Analysis

In the M/G/1+G queue, we have

$$a^*(s) = \lambda(1 - P_{\text{loss}}) \cdot \frac{d^*(s) - a^*_{\text{peak}}(s)}{s}$$

P_{loss}: Loss probability of packets

- $a^*(s)$ is obtained from the following facts:

  (i) The LST $a^*_{\text{peak}}(s)$ of the peak AoI is given in terms of the system delay distribution $D(x)$

    - $D(x) := \Pr(\text{System delay} \le x)$, $\quad d^*(s) = \int_0^\infty e^{-sx} \mathrm{d}D(x)$

  (ii) $D(x)$ and P_{loss} are given by classical results [5] in the queueing theory

[5] Kovalenko, Theory Probab. Appl. (1961).

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Peak AoI in M/G/1+G (1)

Peak AoI = System Delay + Time to Next Update

- We need to consider two exclusive cases

Case 1: There are no packets in the system just after an update

Case 2: There are some packets in the system just after an update

With this observation, we can show that ($h^*(s)$: LST of $H(x)$)

$$a^*_{\text{peak}}(s) = q d^*_0(s) \cdot \frac{\lambda}{s+\lambda} \cdot h^*(s) + (1-q) d^*_+(s) \cdot h^*(s)$$

q: Probability that Case 1 occurs

$d^*_0(s)$: LST of the system delay in Case 1

$d^*_+(s)$: LST of the system delay in Case 2

Peak AoI in M/G/1+G (2)

$$a^*_{\text{peak}}(s) = qd^*_0(s) \cdot \frac{\lambda}{s+\lambda} \cdot h^*(s) + (1-q)d^*_+(s) \cdot h^*(s)$$

- $qd^*_0(s)$ and $(1-q)d^*_+(s)$ are obtained as

$$qd^*_0(s) = \int_0^\infty e^{-sx} e^{-\lambda J(x)} \mathrm{d}D(x), \quad (1-q)d^*_+(s) = d^*(s) - qd^*_0(s),$$

where

$$J(x) = \int_0^x \overline{G}(y)\mathrm{d}y \quad (\overline{G}(x)\text{: CCDF of the deadline distribution})$$

➡ $a^*_{\text{peak}}(s)$ is given in terms of the system delay distribution $D(x)$

Approach of the Analysis

In the M/G/1+G queue, we have

$$a^*(s) = \lambda(1 - P_{\text{loss}}) \cdot \frac{d^*(s) - a^*_{\text{peak}}(s)}{s}$$

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- $a^*(s)$ is obtained from the following facts:

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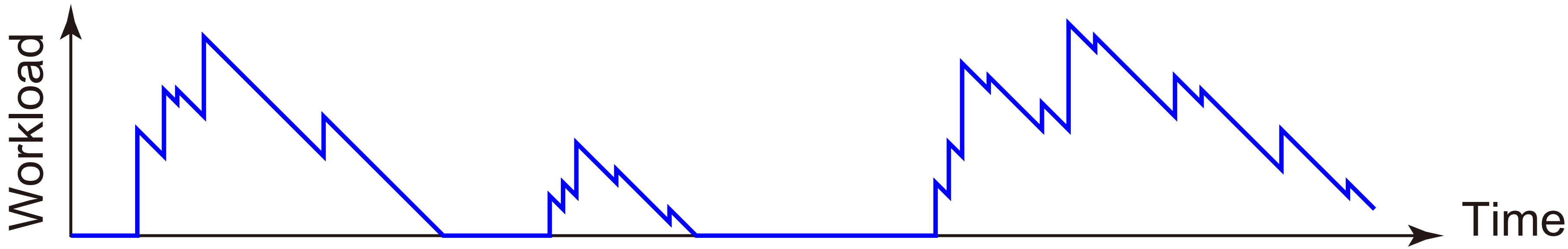
[5] Kovalenko, Theory Probab. Appl. (1961).

Workload process

Workload: Sum of remaining service times of packets which will not be lost (i.e., their deadlines will not expire)

Let V_t denote the workload at time t

- V_t decreases in time linearly with slope 1
- An upward jump in V_t occurs when
  - A packet arrives to the system, and
  - The arriving packet has a deadline larger than V_t just before the arrival instant



Workload Distribution

π_0: Probability that the system is empty
$v(x)$: Probability density function of the workload in system

- The following balance equation holds [5]

$$v(x) = \lambda \pi_0 \overline{H}(x) + \lambda \int_0^x v(y) \overline{G}(y) \overline{H}(x-y) \mathrm{d}y, \quad x \geq 0$$

$\overline{H}(x)$, $\overline{G}(x)$: CCDFs of service times and deadlines

This is a Volterra integral equation of the second kind

➨ $v(x)$ and π_0 are given by solving this integral equation

- In addition, the loss probability P_{loss} is given by

$$P_{\text{loss}} = \int_0^\infty v(x) G(x) \mathrm{d}x = 1 - \frac{1-\pi_0}{\rho}$$

$$\boxed{\rho = \lambda \mathrm{E}[H]}$$

System Delay in M/G/1+G

System Delay = Waiting Time + Service Time

Waiting time = Workload seen by an arriving packet,
conditioned that the packet will not be lost

- $W(0)$: Probability that the waiting time equals to zero
- $w(x)$: Probability density function of the waiting time

$$W(0) = \frac{\pi_0}{1 - P_{\text{loss}}}, \qquad w(x) = \frac{v(x)\overline{G}(x)}{1 - P_{\text{loss}}}$$

The CDF $D(x)$ of the system delay is thus given by

$$D(x) = \frac{\pi_0 H(x)}{1 - P_{\text{loss}}} + \frac{1}{1 - P_{\text{loss}}} \int_0^x H(x-y)\, v(y)\, \overline{G}(y)\, \mathrm{d}y$$

AoI in M/G/1+G

- Finally, substituting the above results into

$$a^*(s) \;=\; \lambda(1-P_{\text{loss}}) \cdot \frac{d^*(s) - a^*_{\text{peak}}(s)}{s},$$

we can obtain the LST of the AoI distribution

$$\boxed{a^*(s) = \pi_0 d_0^*(s) \cdot \frac{\lambda}{s+\lambda} \cdot h^*(s) + (1-\pi_0) d^*(s) \cdot \frac{1-h^*(s)}{s\mathrm{E}[H]}}$$

- $d_0^*(s)$ and $d^*(s)$ are given in terms of
  - π_0: Probability that the system is empty
  - $v(x)$: Probability density function of the workload

(π_0 and $v(x)$ are given by solving the Volterra integral equation)

Special Cases: M/M/1+G and M/M/1+D

Special Case: M/M/1+G (1)

$$a^*(s) = \pi_0 d_0^*(s) \cdot \frac{\lambda}{s+\lambda} \cdot h^*(s) + (1-\pi_0) d^*(s) \cdot \frac{1-h^*(s)}{s\mathrm{E}[H]}$$

($d_0^*(s)$ and $d^*(s)$ are given in terms of π_0 and $v(x)$)

- Service times follow an exponential distribution

$$H(x) = 1 - e^{-\mu x}, \qquad x \geq 0$$

In this case, we have

$$\pi_0 = \left[1 + \int_0^\infty \lambda e^{-\mu x + \lambda J(x)} \mathrm{d}x\right]^{-1}, \qquad v(x) = \pi_0 \lambda e^{-\mu x + \lambda J(x)}$$

$\left(J(x) = \int_0^x \overline{G}(y) \mathrm{d}y, \quad \overline{G}(y)\text{: CCDF of deadlines} \right)$

Special Case: M/M/1+G (2)

$a(x)$: Probability density function of the AoI distribution

$h(x)$ (= $\mu e^{-\mu x}$): Probability density function of service times

- In the M/M/1+G queue, we have

$$\boxed{a(x) = \phi * h(x)}$$

(* denotes the convolution)

➨ The AoI is the sum of two independent random variables

$\phi(x)$ is a probability density function given by

$$\phi(x) = \begin{cases} \dfrac{\pi_0 \mu \lambda (e^{-\lambda x} - e^{-\mu x})}{\mu - \lambda} + \pi_0 \lambda e^{-\mu x + \lambda J(x)}, & \lambda \neq \mu, \\ \pi_0 \mu e^{-\mu x} \left(\mu x + e^{\mu J(x)} \right), & \lambda = \mu \end{cases}$$

Special Case: M/M/1+D (1)

- We further assume that deadlines take a constant value τ
- In this case, $J(x)$ is simplified as

$$J(x) := \int_0^x \overline{G}(x)\mathrm{d}x = \min(x, \tau)$$

➡ We have an explicit formula for the mean AoI

$$\mathrm{E}[A] = \begin{cases} \tau + \dfrac{3-2\rho}{\mu(1-\rho)} + \dfrac{1/\rho - \rho - \mu\tau - 2}{\mu(1-\rho^2 e^{-\mu(1-\rho)\tau})}, & \lambda \neq \mu, \\ \dfrac{2}{\mu} + \dfrac{1}{2\mu} \cdot \dfrac{2+(\mu\tau)^2}{2+\mu\tau}, & \lambda = \mu \end{cases}$$

Special Case: M/M/1+D (2)

Deadlines take a constant value τ

$$\mathrm{E}[A] = \begin{cases} \tau + \dfrac{3-2\rho}{\mu(1-\rho)} + \dfrac{1/\rho - \rho - \mu\tau - 2}{\mu(1-\rho^2 e^{-\mu(1-\rho)\tau})}, & \lambda \neq \mu, \\ \dfrac{2}{\mu} + \dfrac{1}{2\mu} \cdot \dfrac{2+(\mu\tau)^2}{2+\mu\tau}, & \lambda = \mu \end{cases}$$

- We can verify that

$$\lim_{\tau \to 0+} \mathrm{E}[A] = \frac{2}{\mu} + \frac{1}{\rho\mu(1+\rho)}$$ M/M/1/1 queue [6]

$$\lim_{\tau \to \infty} \mathrm{E}[A] = \frac{1}{\mu}\left(1 + \frac{1}{\rho} + \frac{\rho^2}{1-\rho}\right)$$ M/M/1 queue [7]

[6] Costa et al. IEEE Trans. Inf. Theory (2016).

[7] Kaul et al. in Proc. of IEEE INFOCOM 2012 (2012).

Special Case: M/M/1+D (2)

Deadlines take a constant value τ

$$\mathrm{E}[A] = \begin{cases} \tau + \dfrac{3-2\rho}{\mu(1-\rho)} + \dfrac{1/\rho - \rho - \mu\tau - 2}{\mu(1-\rho^2 e^{-\mu(1-\rho)\tau})}, & \lambda \neq \mu, \\ \dfrac{2}{\mu} + \dfrac{1}{2\mu} \cdot \dfrac{2+(\mu\tau)^2}{2+\mu\tau}, & \lambda = \mu \end{cases}$$

- When $\lambda = \mu$ holds, $\mathrm{E}[A]$ is a convex function of τ
  - ➡ $\mathrm{E}[A]$ achieves the minimum at $\tau = \tau^*$

$$\tau^* = \frac{\sqrt{6}-2}{\mu}$$

The minimum value of $\mathrm{E}[A]$ at $\tau = \tau^*$ is given by

$$\mathrm{E}[A] = \frac{\sqrt{6}}{\mu}$$

Conclusion

- We considered the AoI with packet deadline,
  assuming infinite buffer capacity
- We first analyzed the AoI in the M/G/1+G queue
  - The AoI distribution is given in terms of the solution of a Volterra integral equation
- We then considered two special cases
  - For the M/M/1+G queue, we obtained

    Simpler formula for the density function $a(x)$ of the AoI
  - For the M/M/1+D queue, we obtained

    Simpler formula for the mean AoI $\mathrm{E}[A]$