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AoI Perspective on the Accuracy of Monitoring Systems for Continuous-Time Markovian Sources

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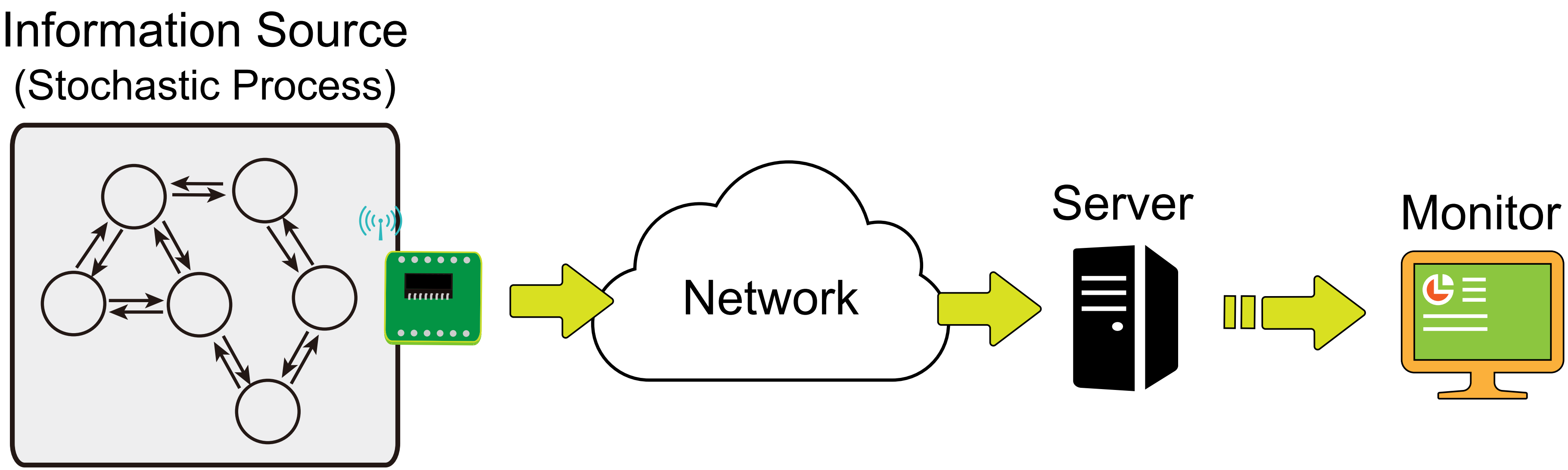
Remote Monitoring System

- The state of an information source is monitored over time



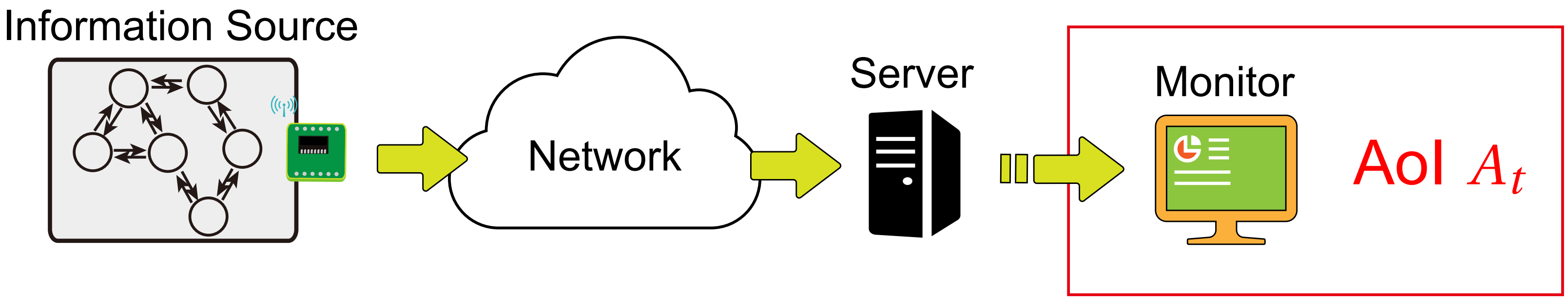
  - The monitor displays the latest state information received

- A_t: Age of Information (AoI) at time t

$$A_t \triangleq t - \eta_t, \quad t \in \mathbb{R}$$

η_t: Time-stamp of the displayed information at time t

AoI and Monitoring Accuracy



- A target value of the AoI

  is highly dependent on the information source dynamics

If the information source is

- slowly varying in time
  - ➨ A fairly large value of the AoI would be acceptable
- quickly varying in time
  - ➨ A strict limit for the AoI would be imposed

Related Works (1)

Only a few works discuss the relation between the AoI and the information source dynamics

- The age of channel state information [Costa et al. (2015)]
  - Modeled as a discrete-time Markov chain with two states
- Optimal samplings of
  - Wiener processes [Sun et al. (2017)]
    - Minimization of the mean squared error
  - Discrete-time Markov chains [Sun and Cyr (2018)]
    - Maximization of the mutual information

Related Works (2)

An optimal sampling of Wiener processes [Sun et al. (2017)]

- The mean squared error

$$\mathrm{MSE} = \mathrm{E}\left[(Y_t - \hat{Y}_t)^2\right]$$

Y_t: Actual state at time t, $\hat{Y}_t$: Displayed state at time t

- A state-dependent sampling policy is optimal
  - An update is generated when $|Y_t - \hat{Y}_t|$ exceeds a threshold
- If sampling timings are independent of the monitored state,

$$\mathrm{MSE} = \mathrm{E}[A]$$

$\mathrm{E}[A]$: Mean AoI

Outline of This Talk

- We consider a monitoring system, where
  - A continuous-time Markov chain $(Y_t)_{t\in\mathbb{R}}$ is monitored
  - The AoI process $(A_t)_{t\in\mathbb{R}}$ is independent of $(Y_t)_{t\in\mathbb{R}}$

- Main contributions:
  - Derive an expression for an accuracy matrix $\boldsymbol{R}$

$$[\boldsymbol{R}]_{i,j} := \Pr\big(\text{Displayed state} = j \mid \text{Actual state} = i\big)$$

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  - Obtain lower and upper bounds for $[\boldsymbol{R}]_{i,i}$

$$[\boldsymbol{R}]_{i,i} = \Pr(\text{Displayed state is correct} \mid \text{Actual state} = i)$$

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    - In particular, we show that $[\boldsymbol{R}]_{i,i} \geq 1 - q_i \mathrm{E}[A]$

q_i: Transition rate of $(Y_t)_{t\in\mathbb{R}}$ from state i

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  - Develop a computational method for $\boldsymbol{R}$ in a special case

Model Description

Continuous-Time Markov Source

- The monitored continuous-time Markov chain $(Y_t)_{t\in\mathbb{R}}$ is
  - stationary ergodic with finite state space $\mathscr{M} = \{1, 2, \ldots, M\}$
  - characterized by transition rates $q_{i,j}$ $(i, j \in \mathscr{M},\ i \neq j)$
    - $q_i := \sum_{j\in\mathscr{M}, j\neq i} q_{i,j}$ Transition rate from i

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    - Sojourn time at state i follows an exponential distribution with mean $1/q_i$
  - ➡ with probability $q_{i,j}/q_i$, a transition to state j occurs

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- $\boldsymbol{Q}$: Infinitesimal generator of $(Y_t)_{t\in\mathbb{R}}$

$$[\boldsymbol{Q}]_{i,j} = \begin{cases} -q_i, & j = i, \\ q_{i,j}, & j \neq i \end{cases}$$

- $\boldsymbol{\pi} = (\pi_1, \pi_2, \dots, \pi_M)$: Stationary probability vector of $(Y_t)_{t\in\mathbb{R}}$

$$\pi_i = \Pr(Y_t = i), \qquad \boldsymbol{\pi}\boldsymbol{Q} = \boldsymbol{0} \quad \text{(Balance equation)}$$

AoI Process and Displayed State

- The AoI process $(A_t)_{t\in\mathbb{R}}$ is assumed to be
  - stationary and ergodic
  - independent of the Markovian information source $(Y_t)_{t\in\mathbb{R}}$
- $\hat{Y}_t$: The displayed state on the monitor at time t

$$\hat{Y}_t = Y_{t-A_t}, \qquad t \in \mathbb{R}$$

We have the following result:

Lemma 1. $(\hat{Y}_t)_{t\in\mathbb{R}}$ is a stationary, ergodic stochastic process with

$$\Pr(\hat{Y}_t = i) = \Pr(Y_t = i) \quad (= \pi_i), \quad i \in \mathcal{M}$$

Main Results

Stochastic Matrices $\boldsymbol{R}$ and $\overline{\boldsymbol{R}}$ (1)

We introduce two matrices $\boldsymbol{R}$ and $\overline{\boldsymbol{R}}$

- The (i, j)th element of $\boldsymbol{R}$ is given by

$$r_{i,j} := \Pr(\text{Displayed state} = j \mid \text{Actual state} = i)$$

- The (i, j)th element of $\overline{\boldsymbol{R}}$ is given by

$$\overline{r}_{i,j} := \Pr(\text{Actual state} = j \mid \text{Displayed state} = i)$$

We can show that the following relation holds w.p. 1

$$r_{i,j} = \lim_{T\to\infty} \frac{\int_{-T/2}^{T/2} \mathbb{1}\{Y_t = i\}\mathbb{1}\left\{\hat{Y}_t = j\right\}\mathrm{d}t}{\int_{-T/2}^{T/2} \mathbb{1}\{Y_t = i\}\mathrm{d}t}$$

$\overline{r}_{i,j}$ also satisfies a similar relation

Stochastic Matrices $\boldsymbol{R}$ and $\overline{\boldsymbol{R}}$ (2)

$$[\boldsymbol{R}]_{i,j} = r_{i,j} := \Pr(\text{Displayed state} = j \mid \text{Actual state} = i)$$

$$[\overline{\boldsymbol{R}}]_{i,j} = \overline{r}_{i,j} := \Pr(\text{Actual state} = j \mid \text{Displayed state} = i)$$

- From Lemma 1 and Baye's formula, we have

$$r_{i,j} = \frac{\pi_j \overline{r}_{j,i}}{\pi_i}, \quad \text{and equivalently,} \quad \boldsymbol{R} = \boldsymbol{\Pi}^{-1} \overline{\boldsymbol{R}}^\top \boldsymbol{\Pi}$$

$$\pi_i = \Pr(Y_t = i) = \Pr(\hat{Y}_t = i), \qquad \boldsymbol{\Pi} = \mathrm{diag}(\pi_1, \pi_2, \ldots, \pi_M)$$

- In particular, we have $r_{i,i} = \overline{r}_{i,i} =: r_i$

$$\begin{aligned} r_i &= \Pr(\text{Displayed state} = i \mid \text{Actual state} = i) \\ &= \Pr(\text{Actual state} = i \mid \text{Displayed state} = i) \end{aligned}$$

r_i is our primary quantity of interest

Formulas of $\boldsymbol{R}$ and $\overline{\boldsymbol{R}}$

$A(x) = \Pr(A_t \le x)$: Probability distribution function of the AoI

- We have the following result:

Theorem 2. $\boldsymbol{R}$ and $\overline{\boldsymbol{R}}$ are given by

$$\boldsymbol{R} = \int_0^\infty \exp\big[(\boldsymbol{\Pi}^{-1}\boldsymbol{Q}^\top\boldsymbol{\Pi})x\big]\mathrm{d}A(x),$$

$$\overline{\boldsymbol{R}} = \int_0^\infty \exp[\boldsymbol{Q}x]\mathrm{d}A(x).$$

$\boldsymbol{Q}$: Infinitesimal generator of $(Y_t)_{t\in\mathbb{R}}$

$\boldsymbol{\Pi} = \mathrm{diag}(\pi_1, \pi_2, \ldots, \pi_M)$

$\boldsymbol{\Pi}^{-1}\boldsymbol{Q}^\top\boldsymbol{\Pi}$ can be regarded as the infinitesimal generator of the time-reversed process $(Y_{-t})_{t\in\mathbb{R}}$

Bounds of r_i (1)

$r_i = \Pr($Displayed state $= i \mid$ Actual state $= i)$

- We can derive the following bounds for r_i:

Theorem 3. r_i is bounded by

$$a^*(q_i) \le r_i \le 1 - (a^*(q_{\max}) - a^*(q_i + q_{\max}))$$

$a^*(s) = \mathrm{E}[e^{-sA_t}]$: LST of the AoI distribution

q_i: Transition rate from state i, $\quad q_{\max} := \max_{i \in \mathcal{M}} q_i$.

- From the inequalities above, we further obtain

Corollary 1. If $\mathrm{E}[A] < \infty$, r_i is bounded by

$$1 - q_i \mathrm{E}[A] \le r_i \le 1 - q_i \mathrm{E}[A] + \frac{(q_i + q_{\max})^2}{2} \mathrm{E}[A^2]$$

Bounds of r_i (2)

$r_i = \Pr(\text{Displayed state} = i \mid \text{Actual state} = i)$

- We have obtained the simple lower bound for r_i:

$$r_i \geq 1 - q_i \mathrm{E}[A]$$

q_i: Transition rate from state i, $\mathrm{E}[A]$: Mean AoI

In order to ensure

$r_i \geq 1 - \epsilon$ for some $\epsilon > 0$,

it is sufficient to design the system so that

$$\mathrm{E}[A] \leq \frac{\epsilon}{q_i}$$

Special Case: Reversible Markovian Information Source

Reversible Markov Chain

- We assume that $(Y_t)_{t\in\mathbb{R}}$ is a reversible Markov chain

$(Y_{-t})_{t\in\mathbb{R}}$ follows the same probability law as $(Y_t)_{t\in\mathbb{R}}$

  - A necessary and sufficient condition:

$$\pi_i q_{i,j} = \pi_j q_{j,i}, \quad i,j \in \mathcal{M}$$

Detailed balance equations

- Under this assumption, we can show that

$\boldsymbol{S} := \boldsymbol{D}\boldsymbol{Q}\boldsymbol{D}^{-1}$ is a symmetric matrix

$$\boldsymbol{D} := \mathrm{diag}(\sqrt{\pi}_1, \sqrt{\pi}_2, \ldots, \sqrt{\pi}_M)$$

Computable Formula of $\boldsymbol{R}$

$\boldsymbol{S} := \boldsymbol{DQD}^{-1}$ is a real symmetric matrix

- $\boldsymbol{S}$ is diagonizable by an orthogonal matrix $\boldsymbol{U} = (\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_M)$

$$\boldsymbol{S} = \sum_{k=1}^{M} \gamma_k \boldsymbol{u}_k \boldsymbol{u}_k^\top$$

γ_k: The kth largest eigenvalue of $\boldsymbol{S}$

$\boldsymbol{u}_k$: A normalized right eigen vector of $\boldsymbol{S}$ associated with γ_k

We can verify that

- $\gamma_k \in \mathbb{R}, \quad k = 1, 2, \ldots, M$
- $\gamma_1 = 0$ and $\gamma_k < 0, \ k = 2, 3, \ldots, M$

- We define $\theta_k := -\gamma_k$ $\quad (0 = \theta_1 < \theta_2 \leq \theta_3 \cdots \leq \theta_M)$

Computable Formula of $\boldsymbol{R}$

- $\boldsymbol{Q}$ is then diagonizable as

$$\boldsymbol{Q} = \sum_{k=1}^{M} (-\theta_k) \boldsymbol{D}^{-1} \boldsymbol{u}_k \boldsymbol{u}_k^{\top} \boldsymbol{D} \qquad \boldsymbol{D} := \mathrm{diag}(\sqrt{\pi}_1, \sqrt{\pi}_2, \ldots, \sqrt{\pi}_M)$$

Recall that $\boldsymbol{R} = \int_0^{\infty} \exp\left[(\boldsymbol{\Pi}^{-1} \boldsymbol{Q}^{\top} \boldsymbol{\Pi}) x\right] \mathrm{d}A(x), \quad \overline{\boldsymbol{R}} = \int_0^{\infty} \exp[\boldsymbol{Q} x] \mathrm{d}A(x)$

We can then show that

$$\boldsymbol{R} = \overline{\boldsymbol{R}} = \boldsymbol{e}\boldsymbol{\pi} + \sum_{k=2}^{M} a^*(\theta_k) \boldsymbol{D}^{-1} \boldsymbol{u}_k \boldsymbol{u}_k^{\top} \boldsymbol{D}$$

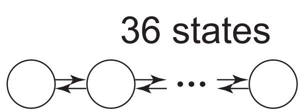
$\boldsymbol{e}$: Column vector with all elements equal to one
$\boldsymbol{\pi}$: Stationary probability vector
$a^*(s)$: LST of the AoI distribution

Numerical Examples

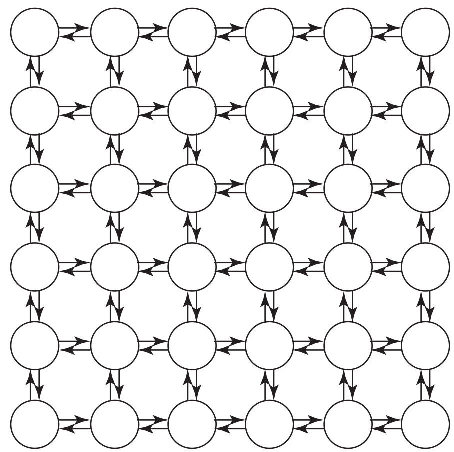
Experimental Setting

- We employ three different Markov chains with 36 states

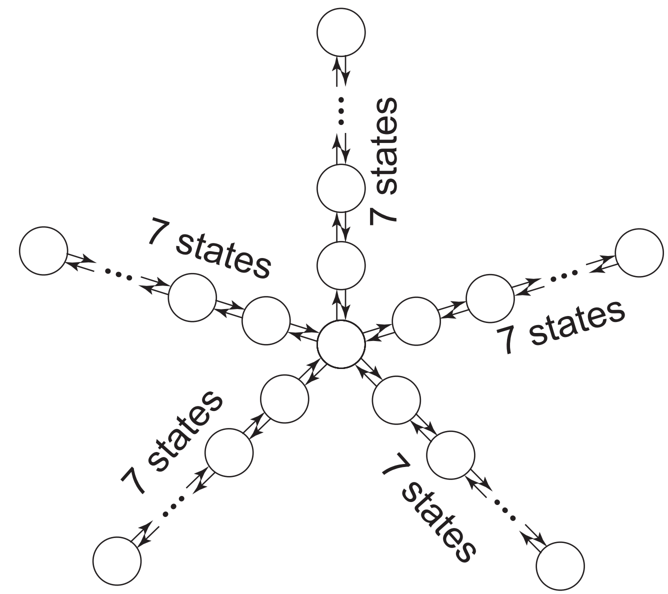
Birth-Death



Grid



Tree

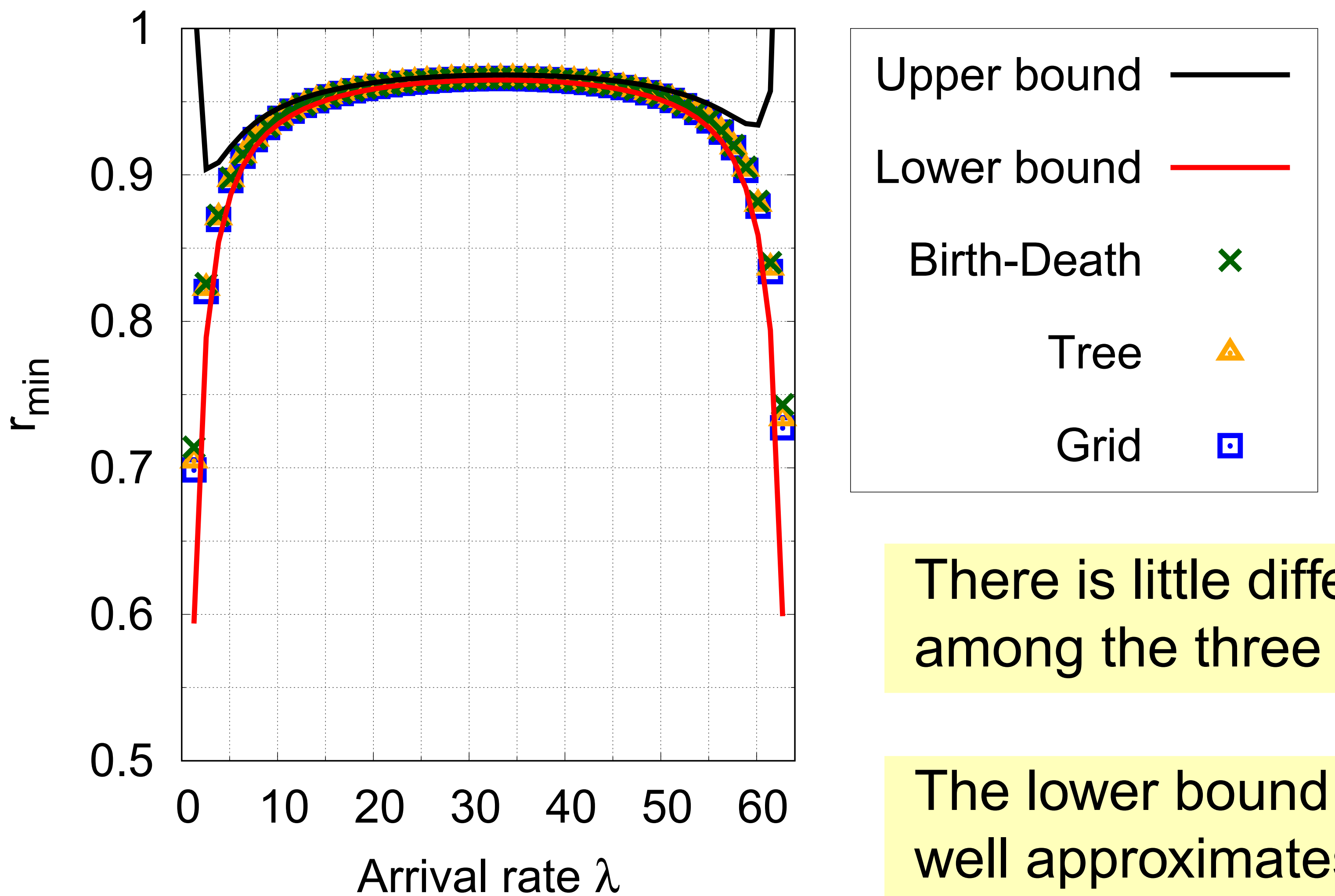


- Transition rates are fixed as $q_i = q \ (i \in \mathcal{M})$
- Transition probabilities are homogeneous

For the monitoring system model, we use an FCFS D/M/1 queue

Numerical Result (1)

$r_{\min} := \min_{i \in \mathcal{M}} r_i$, Transition rate $q = 1$, Service rate $\mu = 64$



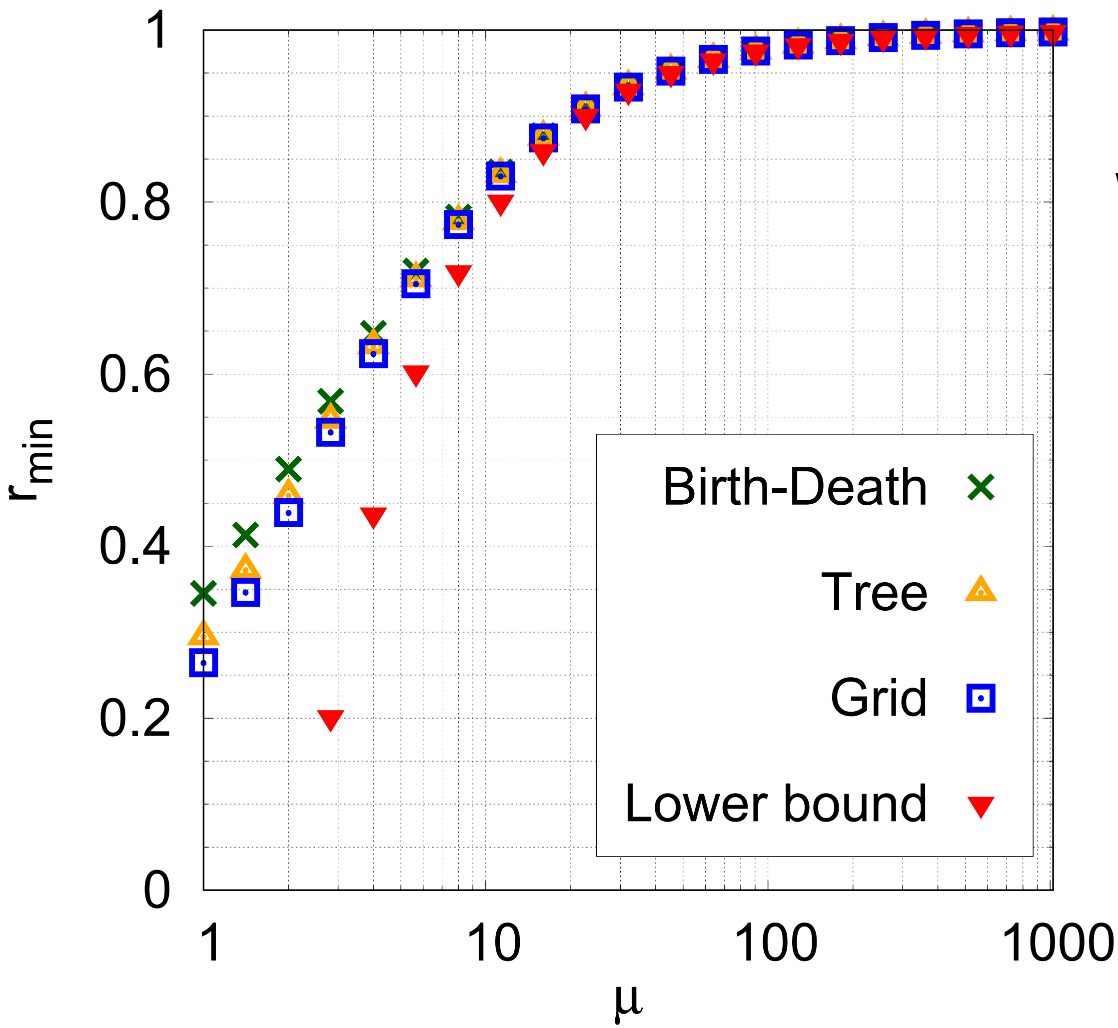
There is little difference in $r_{\min}$ among the three Markov chains

The lower bound $1 - q\mathrm{E}[A]$ well approximates $r_{\min}$

Numerical Result (2)

$r_{\min} := \min_{i \in \mathcal{M}} r_i$, Transition rate $q = 1$

λ for each point is chosen so that $r_{\min}$ is maximized



When μ is large,

- The value of $r_{\min}$ is almost independent of the transition structure
- The lower bound $1 - q\mathrm{E}[A]$ well approximates $r_{\min}$

Conclusion

- We considered a monitoring system, where
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  - The AoI process $(A_t)_{t \in \mathbb{R}}$ is independent of $(Y_t)_{t \in \mathbb{R}}$
- We derived an expression for the accuracy matrix $\boldsymbol{R}$

$$[\boldsymbol{R}]_{i,j} = \Pr\left(\text{Displayed state} = j \mid \text{Actual state} = i\right)$$

- We obtained a simple lower bound for $r_i := [\boldsymbol{R}]_{i,i}$

$r_i \geq 1 - q_i \mathrm{E}[A]$ q_i: Transition rate from state i

- We developed a computing method for $\boldsymbol{R}$ in the reversible case

Through numerical experiments, we observed that the lower bound $1 - q_i \mathrm{E}[A]$ well approximates r_i