Analysis of the Stationary Distribution of the Age of Information

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Joint work with

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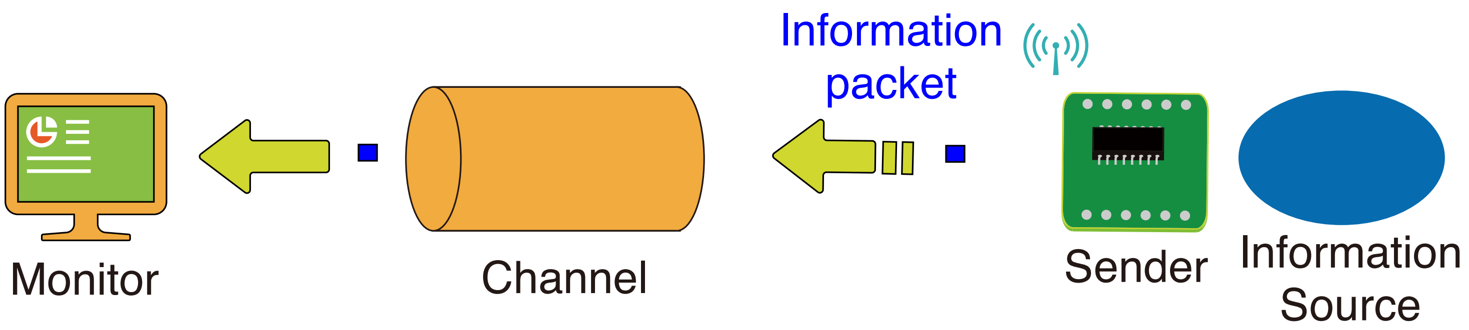
Information Update Systems

A time-varying information source is remotely monitored



- Sender node

  sends observed sample to the monitor

- Receiver node (monitor)

  displays the latest information received

The displayed information is always “older” than the current state

➡ Only partial information can be obtained from the monitor

Information Update Systems

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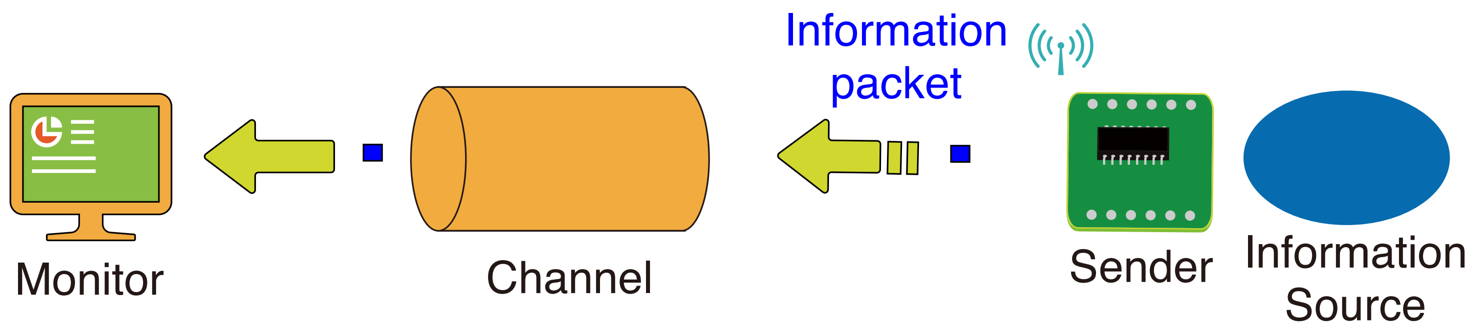

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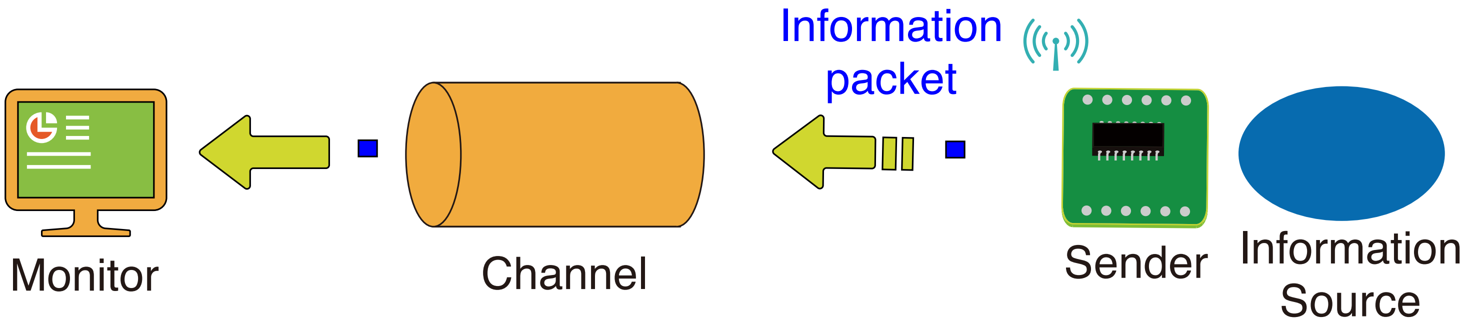

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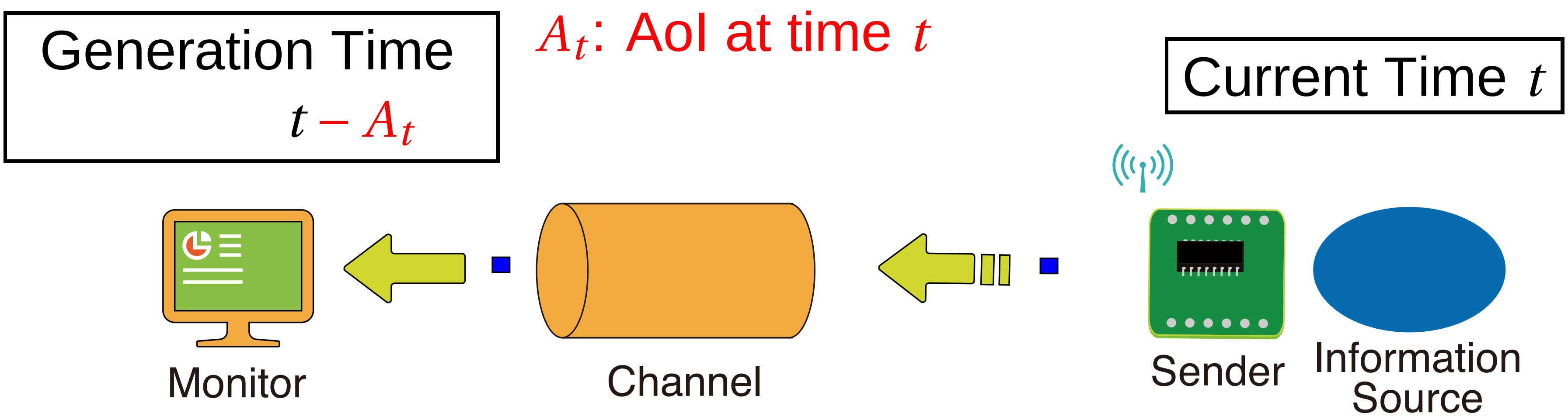
- Age of Information (AoI)

A metric to characterize the freshness of information

Age of Information (AoI)

- AoI: Elapsed time of the information since its generation
  - The smaller the AoI, the fresher the information
- No assumptions on the information source are imposed
  - ➨ The AoI defines “the freshness” in a wide class of systems



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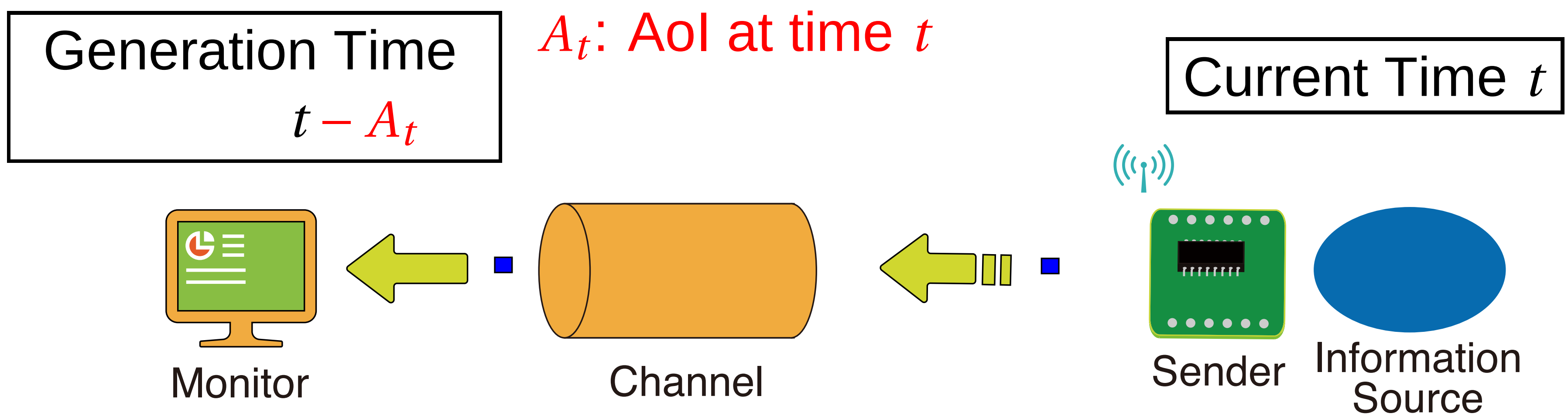


Researches on AoI

Researches on the AoI have been expanding these years

- # of papers in IEEE international conferences
  - 2012: ISIT (1), INFOCOM (1)
  - 2013: ISIT (1)
  - 2014: ISIT (2)
  - 2015: ISIT (3), ICC (2)
  - 2016: ISIT (4), INFOCOM (1), ICC (1)
  - 2017: ISIT (12), INFOCOM (1), ICC (1), GLOBECOM (5)
  - 2018: ISIT (13), INFOCOM (2), ICC (2), GLOBECOM (4)

Outline of This Talk

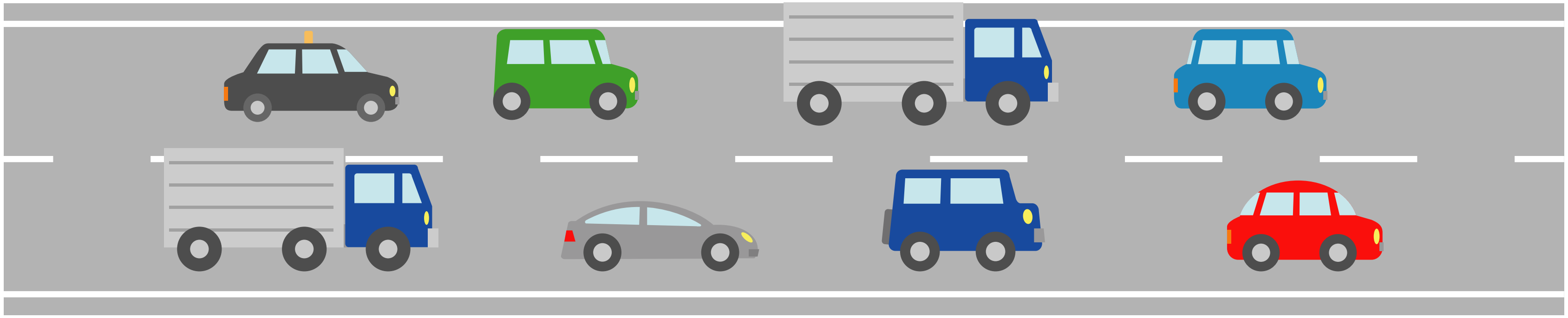
- In the first half, pioneering papers on the AoI are introduced
  - Performance evaluation of VANETs [1]
  - Theoretical analysis based on the queueing theory [2]
- In the second half, our recent work is presented
  - Most previous works discuss only the mean AoI
  - We characterize the probability distribution of the AoI
    - A general formula for the AoI distribution is derived
    - An application to single-server queues is presented

[1] S. Kaul et al., IEEE SECON 2011.
[2] S. Kaul et al., IEEE INFOCOM 2012.

AoI and Queueing Models

Vehiclar Adhoc Networks (VANETs)

- Automobiles are interconnected through wireless channels
- They share each other's information to enhance driving safety
  - Its own position and driving speed
  - Information observed by a sensor and camera
    - Position and speed of neighboring cars
    - Road surface condition



Performance measure in VANETs

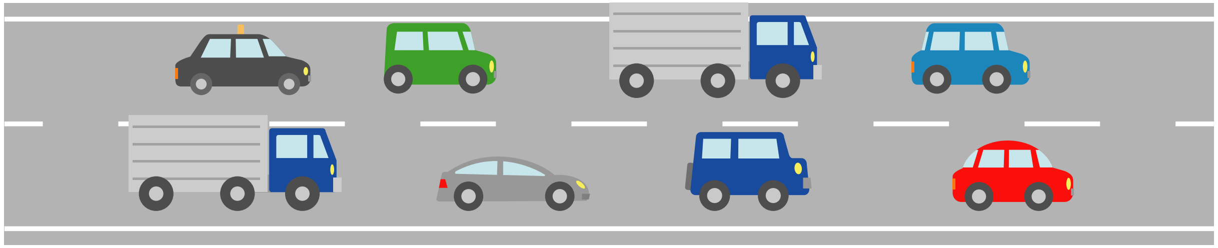
- The situation may change every second
  - The value of information rapidly degenerates

➡ Throughput is not an appropreate performance measure

- Freshness (AoI) is proposed as a performance measure [1]

The ith car's AoI on the jth car $= t - T_{i,j}$ (t: Current time)

- $T_{i,j}$: Time stamp of the last information i received from j



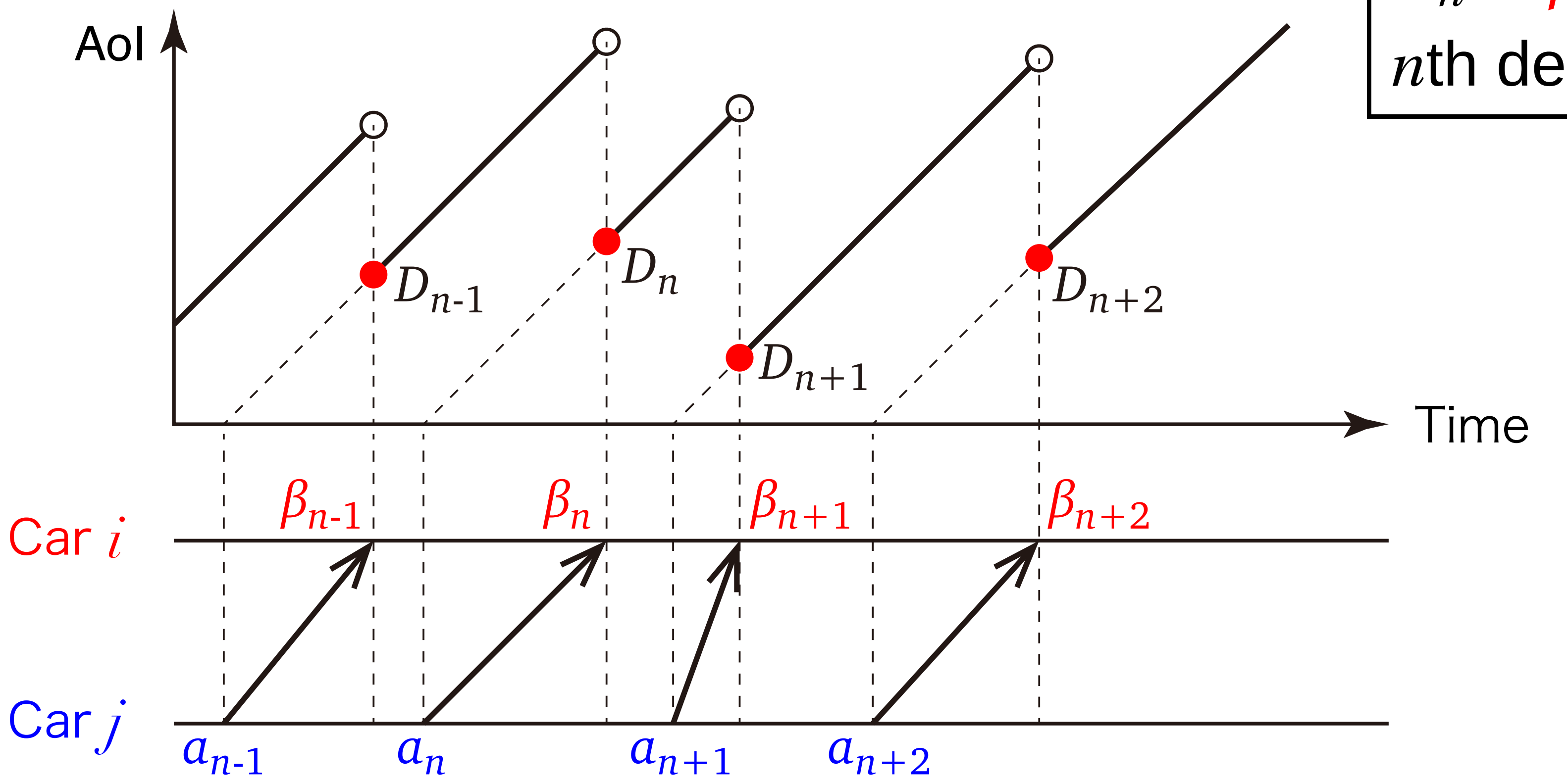
[1] S. Kaul et al., IEEE SECON 2011.

Sample Path of the AoI

Consider a specific pair $(i,\ j)$ of cars

- α_n: Generation time of the nth update
- β_n: Received time of the nth update

$D_n := \beta_n - \alpha_n$
nth delay



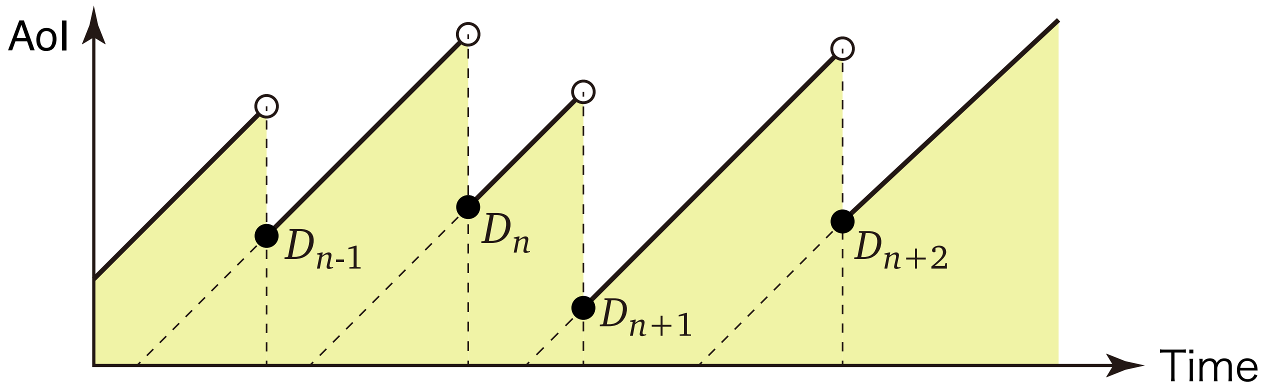
Average AoI

- In [1], the time-average of the AoI (mean AoI)

$$\frac{1}{T}\int_0^T A_t \mathrm{d}t$$ is proposed as a performance measure

A_t: AoI at time t

  - The effect of packet management on the mean AoI is evaluated with simulation experiments

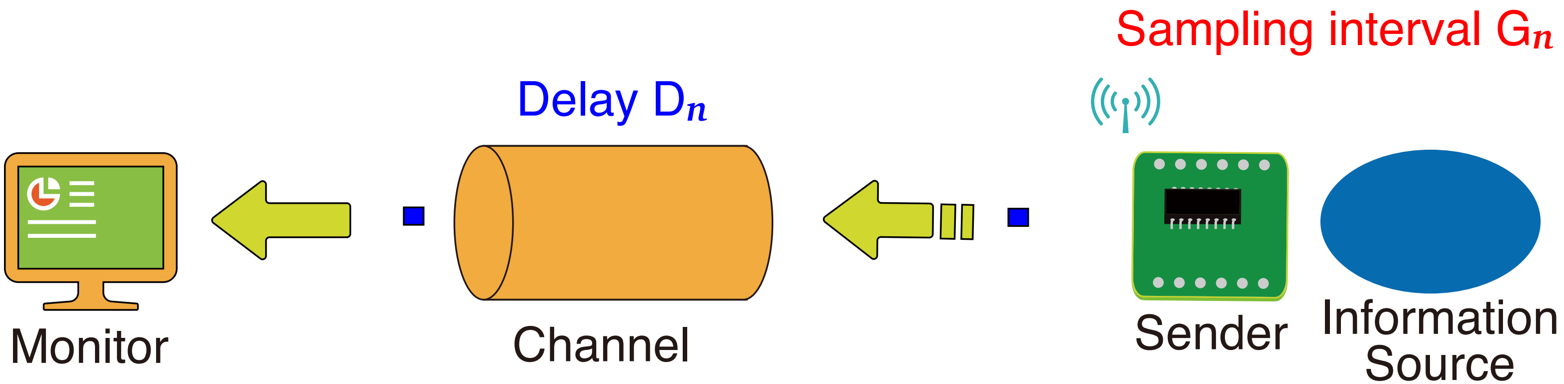


[1] S. Kaul et al., IEEE SECON 2011.

Basic Properties of AoI (1)

Basically, the value of the AoI is determined by

1. Sampling interval G_n
   - If intervals are too large, information updates rarely occur
2. Delay at communication channel D_n
   - If each packet incurs a large delay, the information cannot be kept fresh

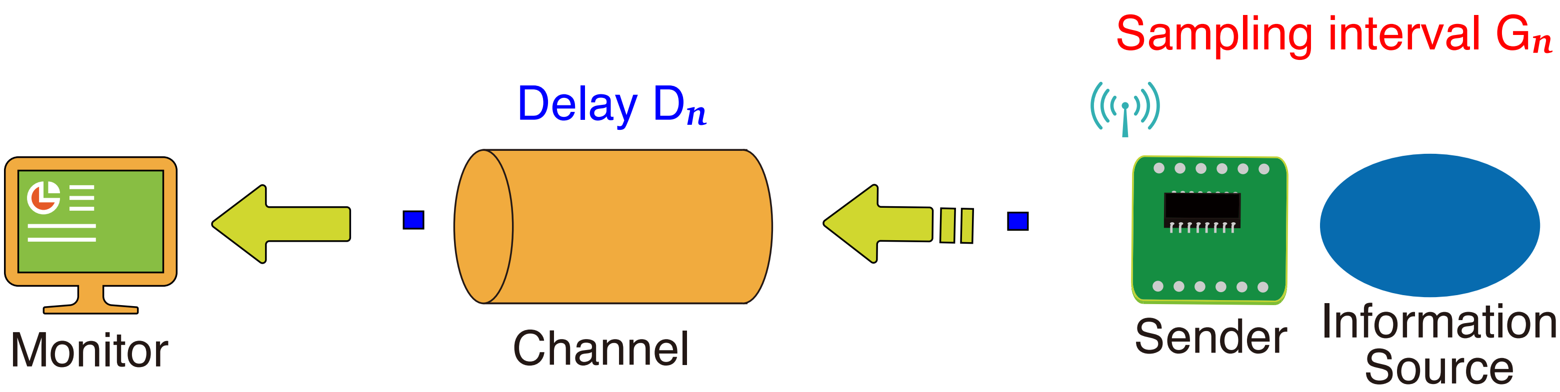


Basic properties of AoI (2)

- In order to make the AoI small,
  1. Sampling interval G_n should be set small
  2. Communication delay D_n should be small
- There is a trade-off between G_n and D_n

  G_n decreases ➨ the traffic load increases ➨ D_n increases

There exists an optimal sampling interval G_n



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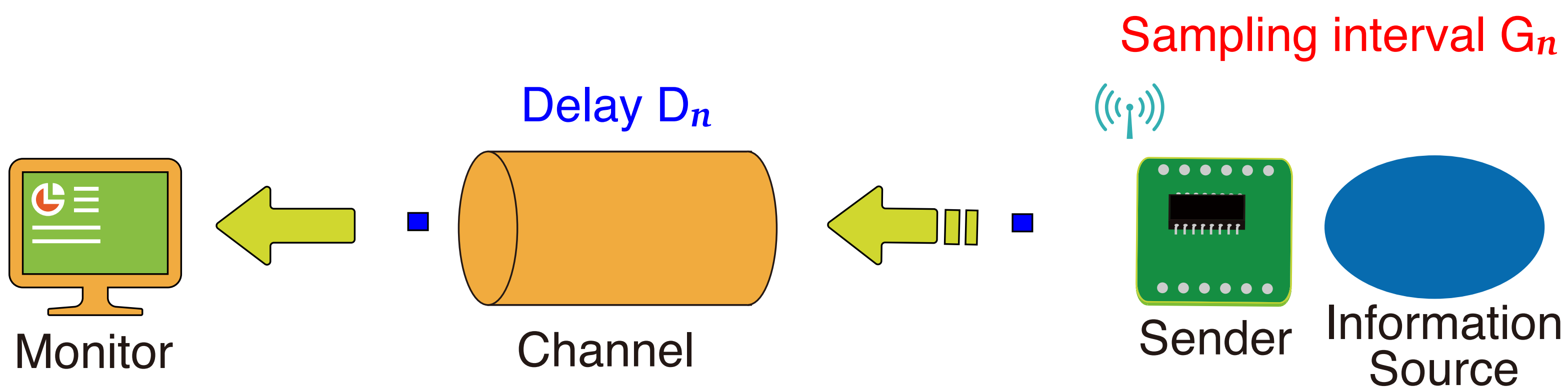
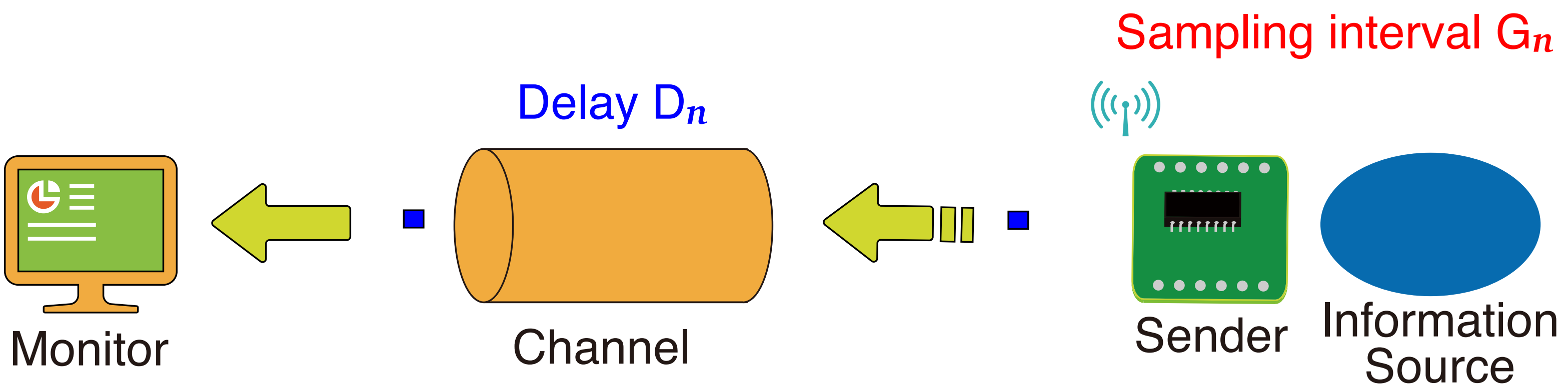


- Formulation of the delay using a queueing model [2]



[2] S. Kaul et al., IEEE INFOCOM 2012.

Queueing Model and AoI

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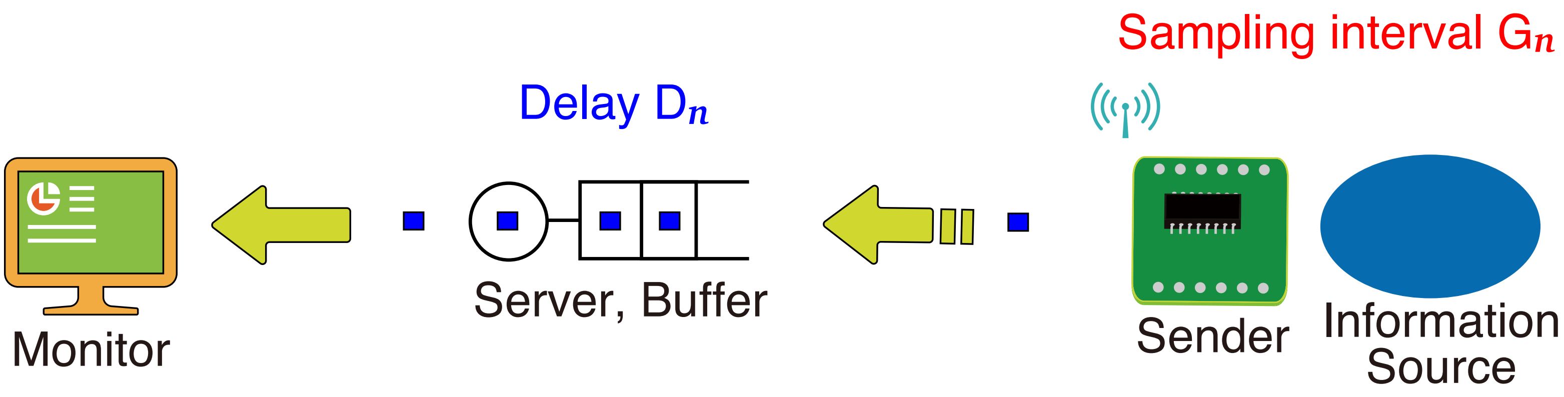

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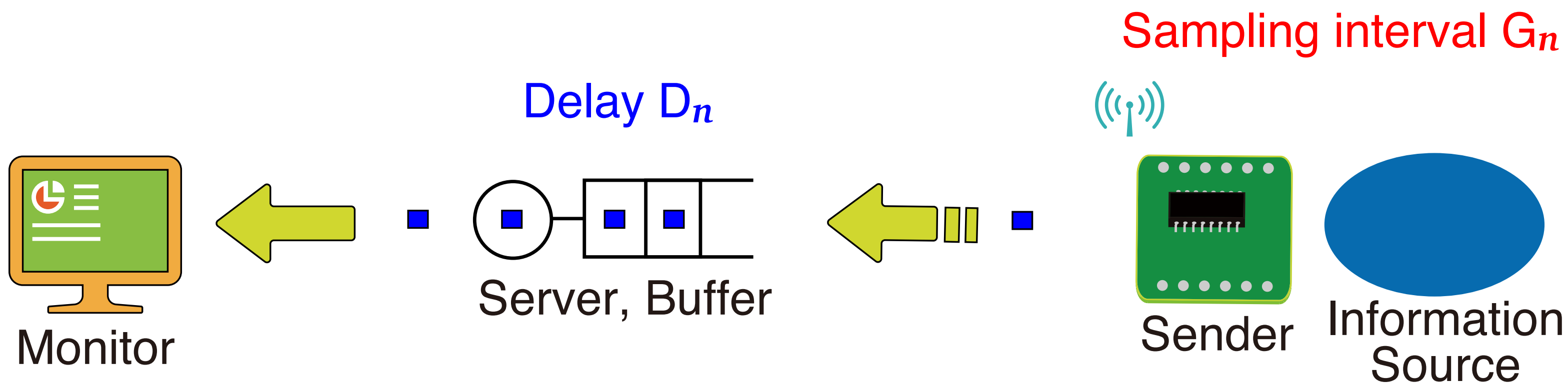
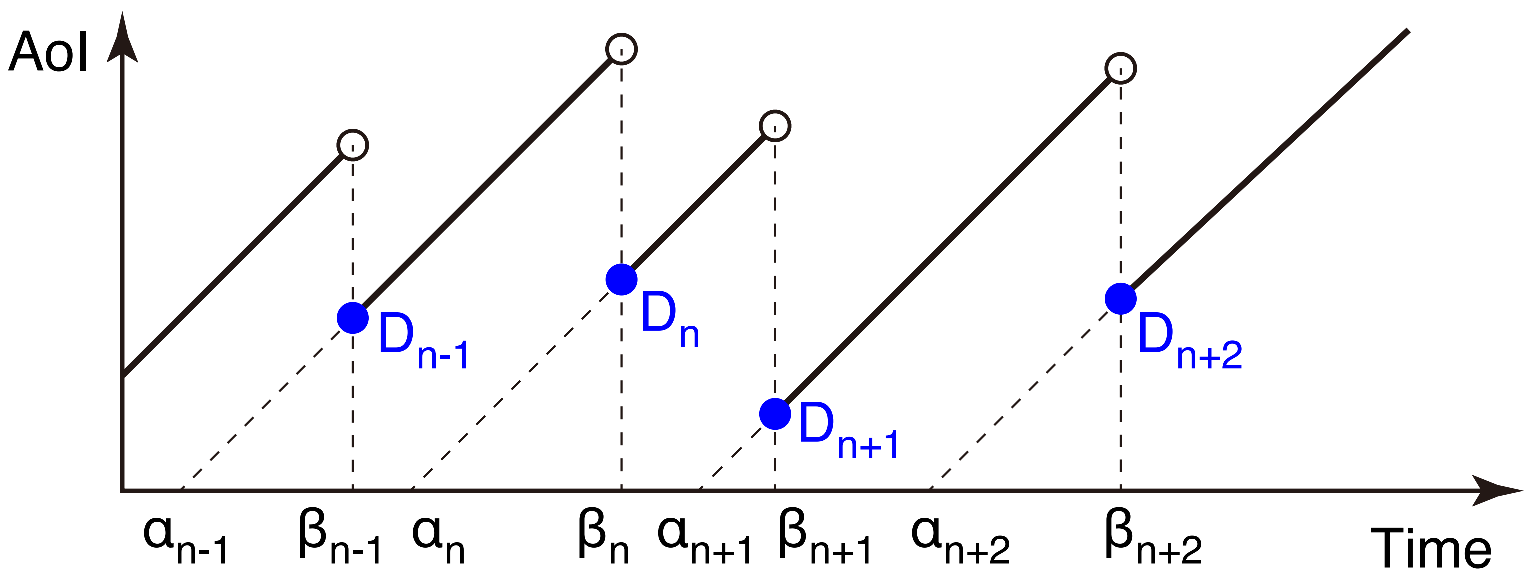
Formula of Mean AoI (1)



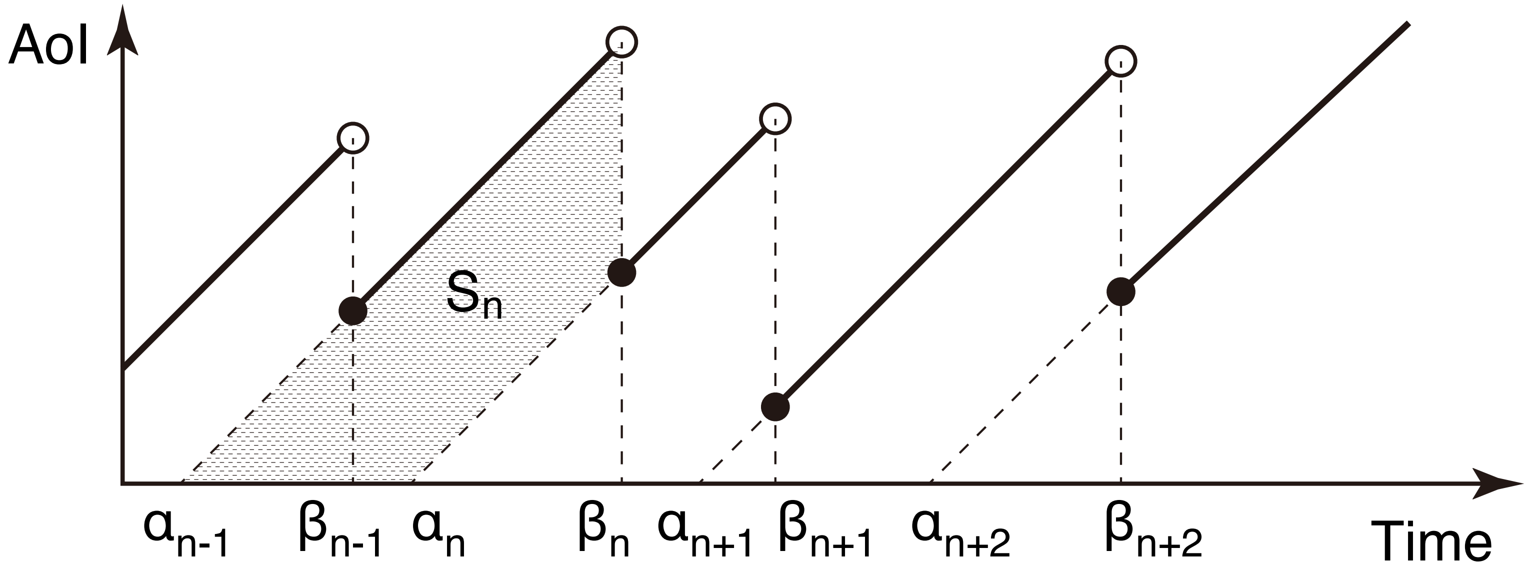
- α_n: Generation time of the nth update
- β_n: Received time of the nth update

$$G_n = \alpha_n - \alpha_{n-1}$$
$$D_n = \beta_n - \alpha_n$$



Formula of Mean AoI (2)

- The mean AoI $\mathrm{E}[A]$ is obtained as follows [2]



$$\mathrm{E}[A] = \lim_{T\to\infty} \frac{1}{T} \sum_{n=0}^{M_T} S_n, \qquad S_n = \frac{(\beta_n - \alpha_{n-1})^2}{2} - \frac{(\beta_n - \alpha_n)^2}{2}$$

M_t: Number of received updates by time t

[2] S. Kaul et al., IEEE INFOCOM 2012.

Formula of Mean AoI (3)

$$\mathrm{E}[A] = \lim_{T\to\infty} \frac{1}{T} \sum_{n=0}^{M_T} S_n, \qquad S_n = \frac{(\beta_n - \alpha_{n-1})^2}{2} - \frac{(\beta_n - \alpha_n)^2}{2}$$

- S_n is rewritten as follows:

$$\begin{aligned} S_n &= \frac{[(\beta_n - \alpha_n) + (\alpha_n - \alpha_{n-1})]^2}{2} - \frac{(\beta_n - \alpha_n)^2}{2} \\ &= \frac{(D_n + G_n)^2 - D_n^2}{2} \\ &= \frac{G_n^2}{2} + G_n D_n \end{aligned}$$

  - G_n: Inter-sampling time between $(n-1)$st and nth updates
  - D_n: Delay of the nth sample

Formula of Mean AoI (4)

G_n: Inter-sampling time between $(n-1)$st and nth updates

D_n: Delay of the nth sample

$$\mathrm{E}[A] = \lim_{T\to\infty} \frac{M_T}{T} \cdot \frac{1}{M_T} \sum_{n=0}^{M_T} S_n, \qquad S_n = \frac{G_n^2}{2} + G_n D_n$$

- If the system is stationary and ergodic, we have [2]

$$\mathrm{E}[A] = \frac{\frac{\mathrm{E}[G^2]}{2} + \mathrm{E}[G_n D_n]}{\mathrm{E}[G]}$$

- In general, G_n and D_n are not independent
  - ➡ Analysis of $\mathrm{E}[A]$ is reduced to derivation of $\mathrm{E}[G_n D_n]$

[2] S. Kaul et al., IEEE INFOCOM 2012.

Formula of Mean AoI (4)

G_n: Inter-sampling time between $(n-1)$st and nth updates

D_n: Delay of the nth sample

$$\mathrm{E}[A] = \lim_{T\to\infty} \frac{M_T}{T} \cdot \frac{1}{M_T} \sum_{n=0}^{M_T} S_n, \qquad S_n = \frac{G_n^2}{2} + G_n D_n$$

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[2] S. Kaul et al., IEEE INFOCOM 2012.

Formula of Mean AoI (5)

In [2], $\mathrm{E}[A]$ are analyzed for three queueing models

$\mathrm{E}[H]$: Mean service time, ρ: Traffic intensity ($= \mathrm{E}[H]/\mathrm{E}[G]$)

(M/M/1) $$\mathrm{E}[A] = \left(1 + \frac{1}{\rho} + \frac{\rho^2}{1-\rho}\right)\mathrm{E}[H]$$

(M/D/1) $$\mathrm{E}[A] = \left(\frac{1}{2} + \frac{1}{2(1-\rho)} + \frac{1-\rho}{\rho e^{-\rho}}\right)\mathrm{E}[H]$$

*This explicit formula is derived in [Y. Inoue et al., IEEE ISIT 2017]

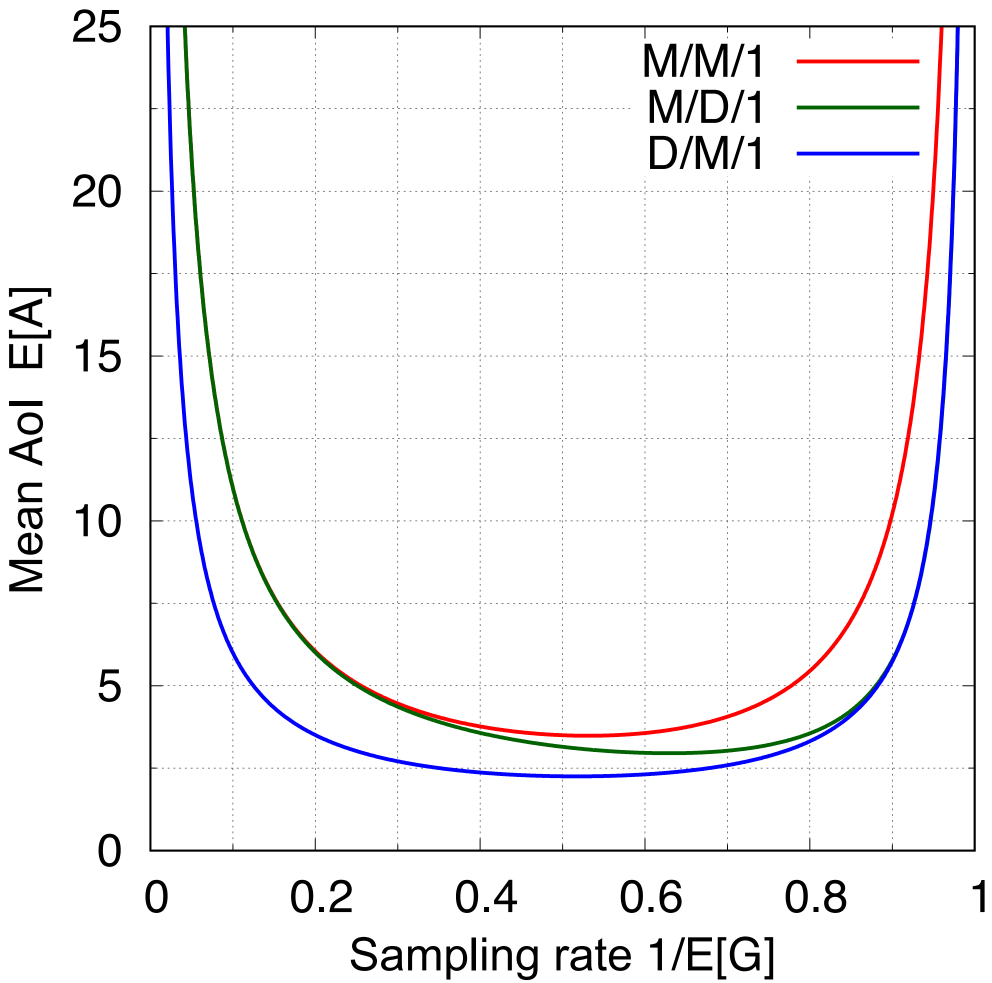
(D/M/1) $$\mathrm{E}[A] = \left(\frac{1}{2\rho} + \frac{1}{1-\gamma}\right)\mathrm{E}[H]$$

γ is the unique solution of $x = e^{-(1-x)/\rho}$ $(0 < x < 1)$

[2] S. Kaul et al., IEEE INFOCOM 2012.

Formula of Mean AoI (6)

- We set the mean service time $\mathrm{E}[H] = 1$



- $\mathrm{E}[A]$ is a U-shaped function of the sampling rate
  - Trade-off between the sampling rate and delay

Constant sampling interval is efficient in terms of $\mathrm{E}[A]$

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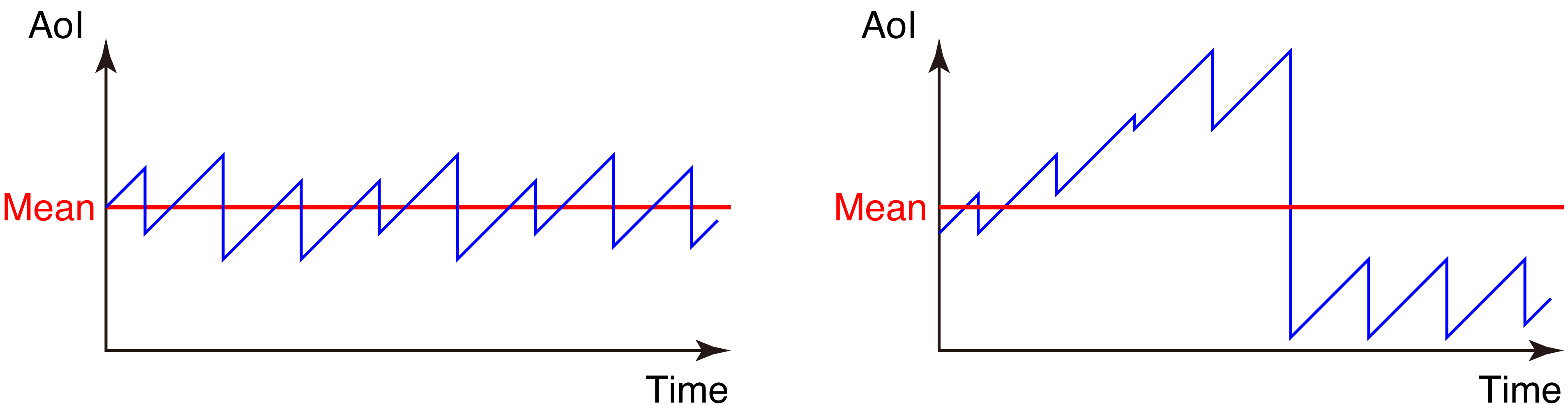
[2] S. Kaul et al., IEEE INFOCOM 2012.

Motivation

- Although $\mathrm{E}[A]$ is a primary performance measure,
  it is not sufficient to characterize the AoI process
  - In particular, the deviation from $\mathrm{E}[A]$ cannot be evaluated



- We are thus interested in the probability distribution of the AoI

$A(x)$: Long-run fraction of time that the AoI $\leq x$

Peak AoI

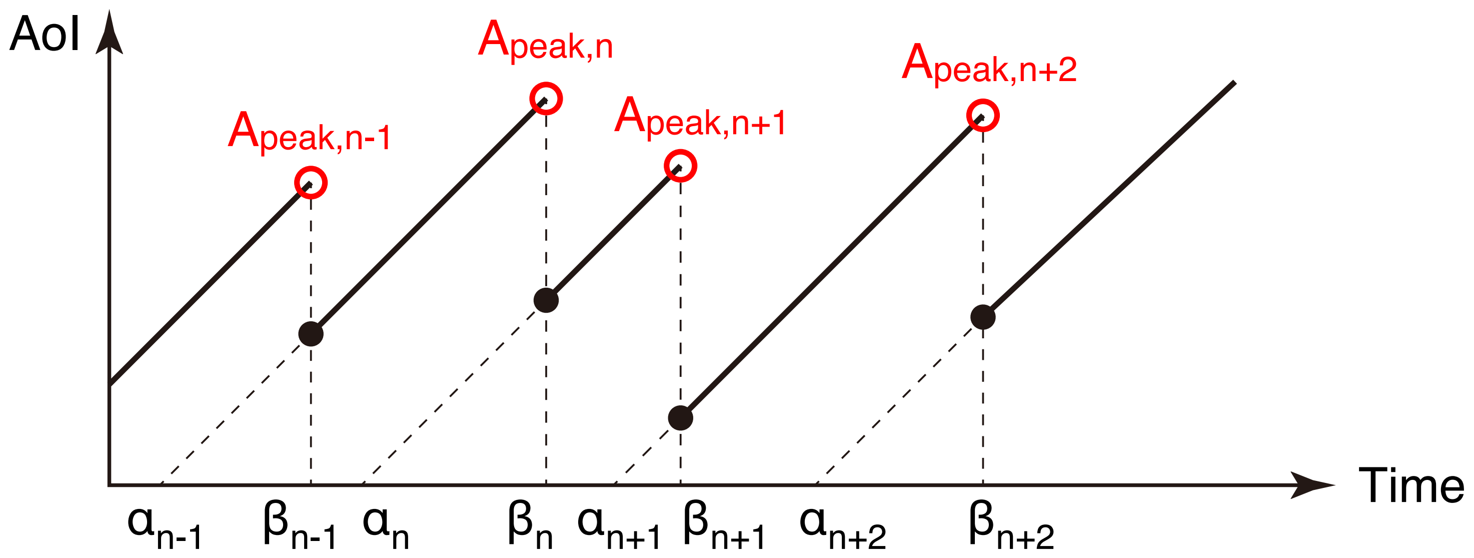
- Probability distribution of the peak AoI is easier to analyze [3]

(≜ AoI just before an update)

$$A_{\text{peak},n+1} = D_n + (\beta_{n+1} - \beta_n)$$

D_n: Delay of the nth packet

β_n: Received time of the nth packet



[3] M. Costa et al., IEEE Trans. Inf. Theory, 62, 2016.

Outline of this work

- We derive a general formula for the AoI distribution $A(x)$:

$$A(x) = \frac{1}{\mathrm{E}[G]} \int_0^x \left(D(y) - A_{\mathrm{peak}}(y)\right) \mathrm{d}y$$

$D(x)$: Delay distribution, $A_{\mathrm{peak}}(x)$: Peak AoI distribution

- Alternative formulas for $\mathrm{E}[A]$ are obtained from this equation

- We present an application to the FCFS GI/GI/1 queue
  - The distribution of the AoI is given in terms of the delay distribution

➡ We specialize this result to the M/GI/1 and GI/M/1 queues

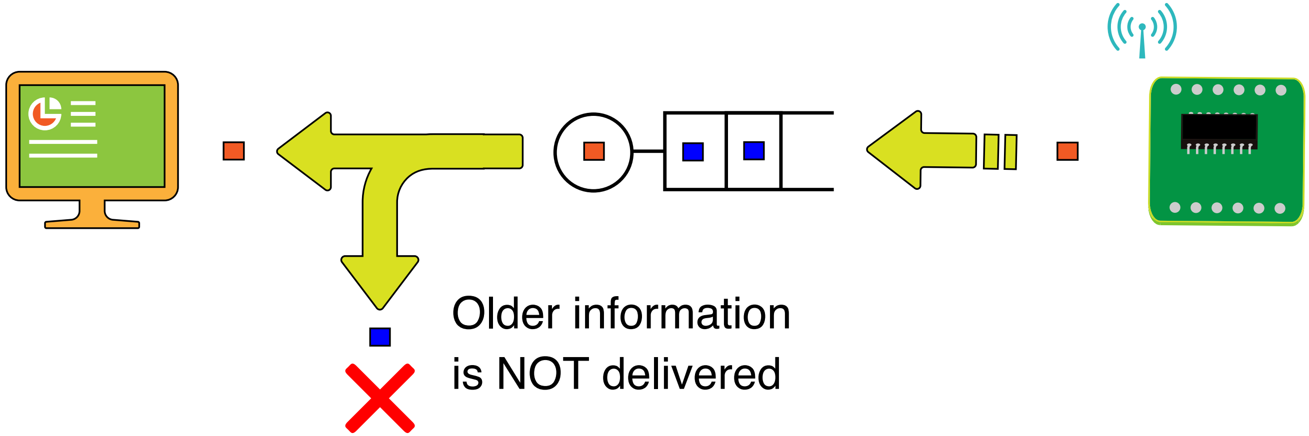
General Formula for the AoI Distribution

Classification of Information Packets

- In general, information packets are classified into two types
  - Informative packets, which contain newer information
  - Non-informative packets, which contain older information

E.g.) FCFS system ➨ All packets are informative

LCFS system ➨ Non-informative packets exist



- If we observe only the stream of informative packets,
  we have a FIFO (First-In-First-Out) queueing system

Sample-Path of General FIFO Queue

We thus consider a general FIFO queue of informative packets

- A sample-path of a general FIFO queue is characterized by

  α_n $(n = 0, 1 \ldots)$: Arrival time of the nth packet

  β_n $(n = 0, 1 \ldots)$: Departure time of the nth packet

  α_n and β_n are deterministic sequences (not random variables)

- We assume the followings

(i)	$\alpha_n \leq \alpha_{n+1}$	(Packets are numbered in order of arrival)
(ii)	$\alpha_n \leq \beta_n$	(A packet cannot depart before its arrival)
(iii)	$\beta_n \leq \beta_{n+1}$	(Packets depart in a FIFO manner)
(iv)	$\alpha_0 \leq \beta_0 = 0 < \alpha_1$	(The system becomes empty at time 0)

AoI and Peak AoI

- M_t: Index of the last departed packet

$$M_t = \sum_{n=1}^{\infty} \mathbb{1}\{\beta_n \leq t\}$$

$$\mathbb{1}\{X\} \triangleq \begin{cases} 1, & X \text{ is true} \\ 0, & X \text{ is false} \end{cases}$$

- A_t: AoI at time t

$$A_t = t - \alpha_{M_t}$$ (Current Time – Time-Stamp)

- $A_{\text{peak},n}$: nth peak AoI

$$A_{\text{peak},n} = \lim_{\Delta t \to 0+} A_{\beta_n - \Delta t}$$ (just before the nth departure)

Asymptotic Frequency Distributions

A_t: AoI at time t $A_{\text{peak},n}$: nth peak AoI

D_n: Delay of the nth packet

- $A^\sharp(x) \triangleq \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbb{1}\{A_t \le x\}\,\mathrm{d}t$ (The fraction of time with $A_t \le x$)
- $A^\sharp_{\text{peak}}(x) \triangleq \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mathbb{1}\left\{A_{\text{peak},n} \le x\right\}$ (The relative number of peak AoIs with $A_{\text{peak},n} \le x$)
- $D^\sharp(x) \triangleq \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mathbb{1}\{D_n \le x\}$ (The relative number of packets with $D_n \le x$)

We assume that these limits exist for each $x \ge 0$

General Formula for the AoI Distribution

$A^\sharp(x)$: AoI distribution, $A^\sharp_{\mathrm{peak}}(x)$: Peak AoI distribution
$D^\sharp(x)$: Delay distribution

- Assumption 1: The arrival rate λ is positive and finite

$$\lambda = \lim_{T\to\infty} \frac{1}{T} \sum_{n=1}^{\infty} \mathbb{1}\{\alpha_n \le T\} \in (0,\infty)$$

- Assumption 2: The system is stable in the sense that

$$\underbrace{\lim_{T\to\infty} \frac{1}{T} \sum_{n=1}^{\infty} \mathbb{1}\left\{\beta_n \le T\right\}}_{\text{Departure rate}} = \underbrace{\lim_{T\to\infty} \frac{1}{T} \sum_{n=1}^{\infty} \mathbb{1}\{\alpha_n \le T\}}_{\text{Arrival rate}} \quad (=\lambda)$$

Under these assumptions, we have

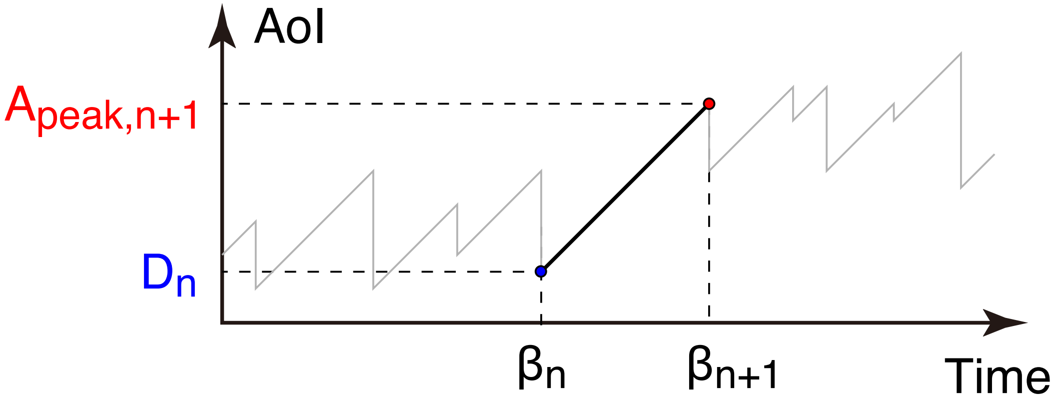
$$A^\sharp(x) = \lambda \int_0^x \left(D^\sharp(y) - A^\sharp_{\mathrm{peak}}(y)\right) \mathrm{d}y$$

Outline of Proof (1)

β_n: Departure time of the nth packet

- In a time interval $t \in [\beta_n, \beta_{n+1})$,
  the AoI linearly increases from D_n to $A_{\text{peak},n+1}$
  - AoI just after an information update: $A_{\beta_n} = D_n$
  - AoI just before an information update: $A_{\beta_{n+1}-} = A_{\text{peak},n+1}$

$$\Rightarrow \int_{\beta_n}^{\beta_{n+1}} \mathbb{1}\{A_t \le x\}\,\mathrm{d}t = \int_{D_n}^{A_{\text{peak},n+1}} \mathbb{1}\{u \le x\}\,\mathrm{d}u$$



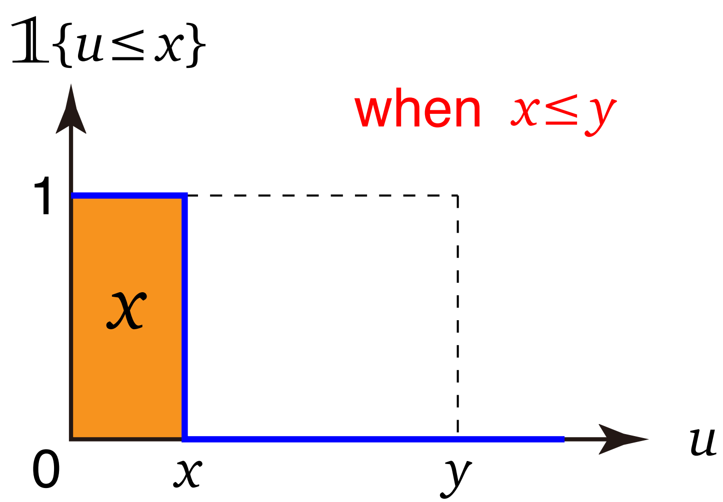
Outline of Proof (2)

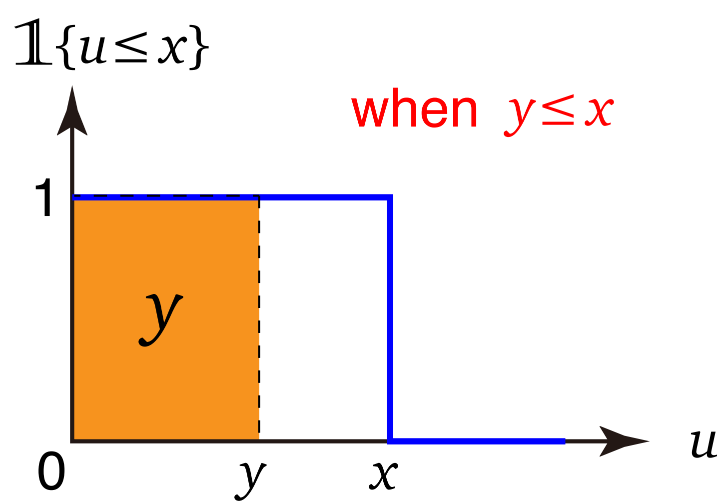
$$\int_{\beta_n}^{\beta_{n+1}} \mathbb{1}\{A_t \le x\}\,\mathrm{d}t = \int_{D_n}^{A_{\mathrm{peak},n+1}} \mathbb{1}\{u \le x\}\,\mathrm{d}u$$

$$= \int_0^{A_{\mathrm{peak},n+1}} \mathbb{1}\{u \le x\}\,\mathrm{d}u - \int_0^{D_n} \mathbb{1}\{u \le x\}\,\mathrm{d}u$$

- For any $x \ge 0$ and $y \ge 0$, we have

$$\int_0^{y} \mathbb{1}\{u \le x\}\,\mathrm{d}u = \min(x, y) = \int_0^{x} \mathbb{1}\{u \le y\}\,\mathrm{d}u$$





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$$= \int_0^{x} \left(1 - \mathbb{1}\{y \le u\}\right)\mathrm{d}u$$

Outline of Proof (2)

$$\int_{\beta_n}^{\beta_{n+1}} \mathbb{1}\{A_t \le x\}\,\mathrm{d}t = \int_{D_n}^{A_{\mathrm{peak},n+1}} \mathbb{1}\{u \le x\}\,\mathrm{d}u$$

$$= \int_0^{A_{\mathrm{peak},n+1}} \mathbb{1}\{u \le x\}\,\mathrm{d}u - \int_0^{D_n} \mathbb{1}\{u \le x\}\,\mathrm{d}u$$

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$$= \int_0^{x} \left(1 - \mathbb{1}\{y \le u\}\right)\mathrm{d}u$$

Therefore, we obtain

$$\int_{\beta_n}^{\beta_{n+1}} \mathbb{1}\{A_t \le x\}\,\mathrm{d}t = \int_0^{x} \mathbb{1}\{D_n \le u\}\,\mathrm{d}u - \int_0^{x} \mathbb{1}\{A_{\mathrm{peak},n+1} \le u\}\,\mathrm{d}u$$

Outline of Proof (3)

$$\int_{\beta_n}^{\beta_{n+1}} \mathbb{1}\{A_t \le x\}\,\mathrm{d}t = \int_0^x \mathbb{1}\{D_n \le u\}\,\mathrm{d}u - \int_0^x \mathbb{1}\left\{A_{\mathrm{peak},n+1} \le u\right\}\mathrm{d}u$$

- The distribution $A^\sharp(x)$ of the AoI is thus given by

$$\begin{aligned}
A^\sharp(x) &= \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbb{1}\{A_t \le x\}\,\mathrm{d}t \\
&= \lim_{T\to\infty} \frac{1}{T}\sum_{n=0}^{M_T}\int_{\beta_n}^{\beta_{n+1}} \mathbb{1}\{A_t \le x\}\,\mathrm{d}t \\
&= \lim_{T\to\infty} \frac{M_T}{T}\cdot\frac{1}{M_T}\sum_{n=0}^{M_T}\left[\int_0^x \mathbb{1}\{D_n \le u\}\,\mathrm{d}u - \int_0^x \mathbb{1}\left\{A_{\mathrm{peak},n+1} \le u\right\}\mathrm{d}u\right] \\
&= \lambda\int_0^x \left(D^\sharp(y) - A^\sharp_{\mathrm{peak}}(y)\right)\mathrm{d}y
\end{aligned}$$

Stationarity and Ergodicity

$$A^{\sharp}(x) = \lambda \int_0^x \left(D^{\sharp}(y) - A^{\sharp}_{\mathrm{peak}}(y) \right) \mathrm{d}y$$ holds sample-path-wise

- We assume that the system is stationary and ergodic
  - Probability distributions of system-states are time-invariant
    - They are called stationary distributions
  - Stationary distributions = Probability distributions on a sample-path

➡ $A(x) = A^{\sharp}(x),\ A_{\mathrm{peak}}(x) = A^{\sharp}_{\mathrm{peak}}(x),$ and $D(x) = D^{\sharp}(x)$

$A(x)$: Stationary AoI distribution
$A_{\mathrm{peak}}(x)$: Stationary peak AoI distribution
$D(x)$: Stationary delay distribution

Notation

We use the following convention throughout the discussion below

- For any non-negative random variable F,
  - $F(x)$: Probability distribution function of F

    $$\Pr(F \leq x) = F(x)$$

  - $f(x)$: Probability density function of F

    $$f(x) = \frac{d}{dx}F(x), \qquad x \geq 0$$

  - $f^*(s)$: Laplace-Stieltjes transform (LST) of F

    $$f^*(s) = \mathrm{E}\left[e^{-sF}\right] = \int_0^\infty e^{-sx}\mathrm{d}F(x), \quad \mathrm{Re}(s) > 0$$

Stationary Distribution of the AoI (1)

A: AoI, A_{peak}: Peak AoI, D: Delay

- The density function of the stationary AoI

$$a(x) = \frac{D(x) - A_{\text{peak}}(x)}{\mathrm{E}[G]}$$

- The LST of the stationary AoI

$$a^*(s) = \frac{d^*(s) - a^*_{\text{peak}}(s)}{s\mathrm{E}[G]}$$

- The kth moment of the stationary AoI ($k = 1, 2, \ldots$)

$$\mathrm{E}[A^k] = \frac{\mathrm{E}[(A_{\text{peak}})^{k+1}] - \mathrm{E}[D^{k+1}]}{(k+1)\mathrm{E}[G]}$$

Formulas for the Mean AoI

- The formula for the mean AoI $\mathrm{E}[A]$ [1]

(i) $$\mathrm{E}[A] = \frac{\frac{\mathrm{E}[G^2]}{2} + \mathrm{E}[G_n D_n]}{\mathrm{E}[G]}$$

- We have the following alternative formulas for $\mathrm{E}[A]$:

(ii) $$\mathrm{E}[A] = \frac{E[(A_{\mathrm{peak}})^2] - \mathrm{E}[D^2]}{2\mathrm{E}[G]}$$

(iii) $$\mathrm{E}[A] = \lim_{s \to 0+} (-1) \cdot \frac{\mathrm{d}}{\mathrm{d}s} \left[a^*(s) \right], \qquad a^*(s) = \frac{d^*(s) - a^*_{\mathrm{peak}}(s)}{s\mathrm{E}[G]}$$

[1] S. Kaul et al., in Proc. of IEEE INFOCOM 2012, 2731–2735, Mar. 2012.

Stationary Distribution of the AoI (2)

- The distribution $A(x)$ of the AoI is given by

$$A(x) = \lambda \int_0^x \left(D(y) - A_{\mathrm{peak}}(y) \right) \mathrm{d}y$$

  - The Delay is widely analyzed in the queueing theory
  - We need an additional analysis on the peak AoI

- Below, we consider FCFS single-server queues
  - In the GI/GI/1 queue, $A_{\mathrm{peak}}(x)$ is given in terms of $D(x)$
  - In the M/GI/1 and GI/M/1 queues, we can obtain formulas of the AoI from the known results for $D(x)$

Application to the FCFS GI/GI/1 Queue

GI/GI/1 Queue

- Inter-arrival times are assumed to be i.i.d. with
  - general probability distribution function $G(x)$
  - mean $\mathrm{E}[G]$ ($\lambda = 1/\mathrm{E}[G]$ follows)
- Processing times are assumed to be i.i.d. with
  - general probability distribution function $H(x)$
  - mean $\mathrm{E}[H]$
- The traffic intensity $\rho \triangleq \lambda \mathrm{E}[H]$
  - We assume $\rho < 1$ so that the system is stable
  - We also assume that the system is stationary and ergodic

Peak AoI Distribution

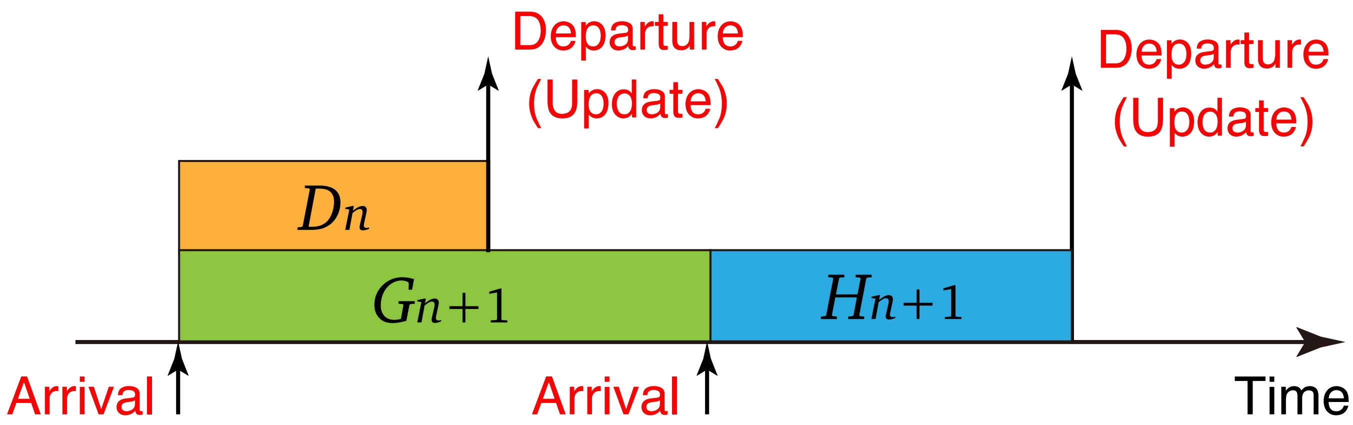
G_n: Inter-arrival time between the $(n-1)$st and nth packets

H_n: The service time of the nth packet

D_n: The delay of the nth packet

- If $G_{n+1} > D_n$ (The system becomes empty on departure)

$$A_{\text{peak},n+1} = G_{n+1} + H_{n+1}$$



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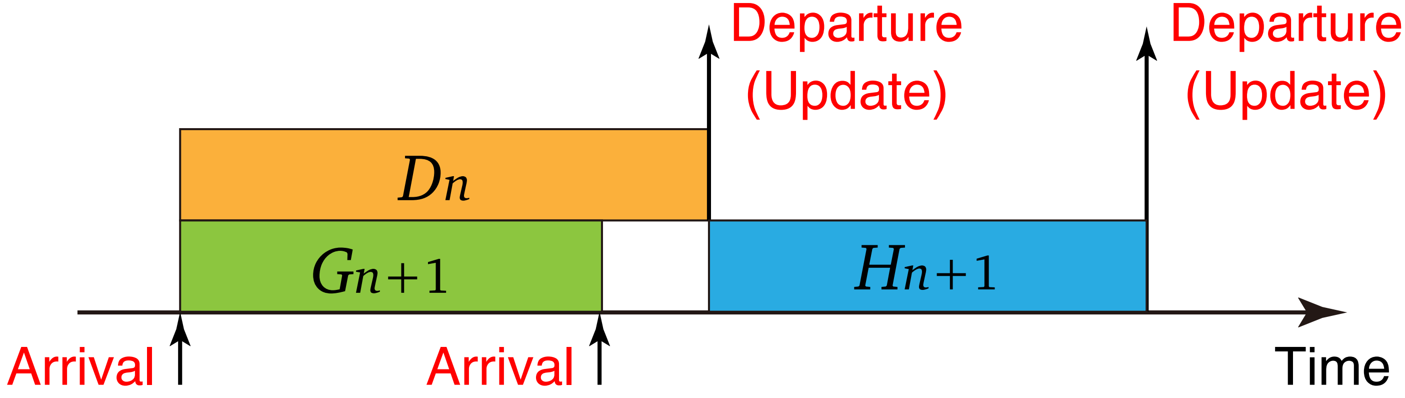
$$A_{\text{peak},n+1} = G_{n+1} + H_{n+1}$$

- If $G_{n+1} \le D_n$ (The next service starts just after departure)

$$A_{\text{peak},n+1} = D_n + H_{n+1}$$



Peak AoI Distribution

G_n: Inter-arrival time between the $(n-1)$st and nth packets

H_n: The service time of the nth packet

D_n: The delay of the nth packet

- If $G_{n+1} > D_n$ (The system becomes empty on departure)

$$A_{\text{peak},n+1} = G_{n+1} + H_{n+1}$$

- If $G_{n+1} \leq D_n$ (The next service starts just after departure)

$$A_{\text{peak},n+1} = D_n + H_{n+1}$$

We thus obtain

$$A_{\text{peak},n+1} = \max(G_{n+1}, D_n) + H_{n+1}$$

Peak AoI Distribution (2)

G_n: Inter-arrival time between the $(n-1)$st and nth packets
H_n: Service time of the nth packet
D_n: Delay of the nth packet

- $A_{\text{peak},n} = \max(G_{n+1}, D_n) + H_{n+1}$
- G_{n+1}, D_n, and H_{n+1} are independent
  - ➨ We obtain the LST of the stationary peak AoI

$$a^*_{\text{peak}}(s) = \left[\int_0^\infty e^{-sx} G(x) \mathrm{d}D(x) + \int_0^\infty e^{-sx} D(x) \mathrm{d}G(x) - \mathrm{E}\left[\mathbb{1}\{G = D\} e^{-sG}\right]\right] h^*(s)$$

Summary of Results (GI/GI/1 Queue)

- The LST of the stationary AoI

$$a^*(s) = \frac{d^*(s) - a^*_{\mathrm{peak}}(s)}{s\mathrm{E}[G]}$$

- The LST of the stationary peak AoI

$$a^*_{\mathrm{peak}}(s) = \left[\int_0^\infty e^{-sx} G(x)\mathrm{d}D(x) + \int_0^\infty e^{-sx} D(x)\mathrm{d}G(x) - \mathrm{E}\left[\mathbb{1}\{G = D\}\, e^{-sG}\right]\right] h^*(s)$$

- $a^*(s)$ is given in terms of the delay distribution

Special Cases: M/GI/1 and GI/M/1 Queues

M/GI/1 Queue (1)

- Exponential inter-arrival time distribution $G(x) = 1 - e^{-\lambda x}$
- General service time distribution $H(x)$ (LST $h^*(s)$)
    - M/M/1 and M/D/1 [2] are special cases of this model

In this model, the LST of the system delay D is given by

$$d^*(s) = \frac{(1-\rho)s}{s - \lambda + \lambda h^*(s)} \cdot h^*(s)$$

➨ We obtain the LST of the stationary AoI distribution

$$a^*(s) = \rho d^*(s) \cdot \frac{1 - h^*(s)}{\mathrm{E}[H]s} + d^*(s+\lambda) \cdot \frac{\lambda}{s+\lambda} \cdot h^*(s)$$

[2] S. Kaul et al., IEEE INFOCOM 2012.

M/GI/1 Queue (2)

- The first two moments of the AoI distribution are given by

$$\mathrm{E}[A] = \mathrm{E}[D] + \frac{1-\rho}{\rho h^*(\lambda)} \cdot \mathrm{E}[H]$$

$$\mathrm{E}[A^2] = \mathrm{E}[D^2] + \frac{2(1-\rho)}{(\rho h^*(\lambda))^2} \left[1 + \rho h^*(\lambda) + \lambda h^{(1)}(\lambda)\right] (\mathrm{E}[H])^2$$

where

$$\mathrm{E}[D] = \frac{\lambda \mathrm{E}[H^2]}{2(1-\rho)} + \mathrm{E}[H]$$

$$\mathrm{E}[D^2] = \frac{\lambda \mathrm{E}[H^3]}{3(1-\rho)} + \frac{(\lambda \mathrm{E}[H^2])^2}{2(1-\rho)^2} + \frac{\mathrm{E}[H^2]}{1-\rho}$$

GI/M/1 Queue (1)

- General inter-arrival time distribution $G(x)$ (LST $g^*(s)$)
- Exponential processing time distribution $H(x) = 1 - e^{-\mu x}$
  - D/M/1 queue [2] is a special case of this model

In this model, the system delay distribution is exponential

$$D(x) = 1 - e^{-(1-\gamma)\mu x}, \qquad d^*(s) = \frac{(1-\gamma)\mu}{s + (1-\gamma)\mu}$$

where γ is a unique solution of $\gamma = g^*(\mu - \mu\gamma)$ and $\gamma \in (0,1)$

➡ We obtain the LST of the stationary AoI distribution

$$a^*(s) = \left[\rho \cdot \frac{(1-\gamma)\mu}{s + (1-\gamma)\mu} \cdot \frac{g^*(s + (1-\gamma)\mu) - \gamma}{1-\gamma} + \frac{1 - g^*(s)}{s\mathrm{E}[G]}\right] \frac{\mu}{s+\mu}$$

[2] S. Kaul et al., IEEE INFOCOM 2012.

GI/M/1 Queue (2)

- The first two moments of the AoI distribution are given by

$$\mathrm{E}[A] = \frac{\mathrm{E}[G^2]}{2\mathrm{E}[G]} + \frac{1}{\mu} - \frac{g^{(1)}((1-\gamma)\mu)}{(1-\gamma)\mu\mathrm{E}[G]}$$

$$\mathrm{E}[A^2] = \frac{\mathrm{E}[G^3]}{3\mathrm{E}[G]} + \rho\mathrm{E}[G^2] + \frac{2}{\mu^2} + \frac{\rho}{1-\gamma}\left[g^{(2)}((1-\gamma)\mu) - 2\left(\frac{1}{(1-\gamma)\mu} + \frac{1}{\mu}\right)g^{(1)}((1-\gamma)\mu)\right]$$

M/M/1 Queue

- Exponential inter-arrival and service times

$$G(x) = 1 - e^{-\lambda x}, \qquad H(x) = 1 - e^{-\mu x}$$

- The LST of the stationary AoI is simplified as

$$a^*(s) = \frac{(1-\rho)\mu}{s + (1-\rho)\mu} - \frac{(1-\rho)\mu s(s+\lambda+\mu)}{(s+\mu)^2(s+\lambda)}$$

➡ We obtain an explicit formula for the AoI distribution $A(x)$

$$A(x) = 1 - e^{-(1-\rho)\mu x} + \left(\frac{1}{1-\rho} + \rho\mu x\right) e^{-\mu x} - \frac{1}{1-\rho} \cdot e^{-\lambda x}$$

Summary

- The age of information (AoI) has been attracting a considerable attention of reseach communities
  - The AoI represents the freshness of information
  - Most previous works focus only on the mean AoI

In our work,

- A general formula for the AoI distribution was derived

$$A(x) = \lambda \int_0^x \big(D(y) - A_{\mathrm{peak}}(y)\big)\,\mathrm{d}y, \quad x \geq 0$$

- We presented an application to single-server FCFS queues
- A full paper including results for LCFS queues is available at https://arxiv.org/abs/1804.06139