The Probability Distribution of the AoI in Queues with Infinitely Many Servers

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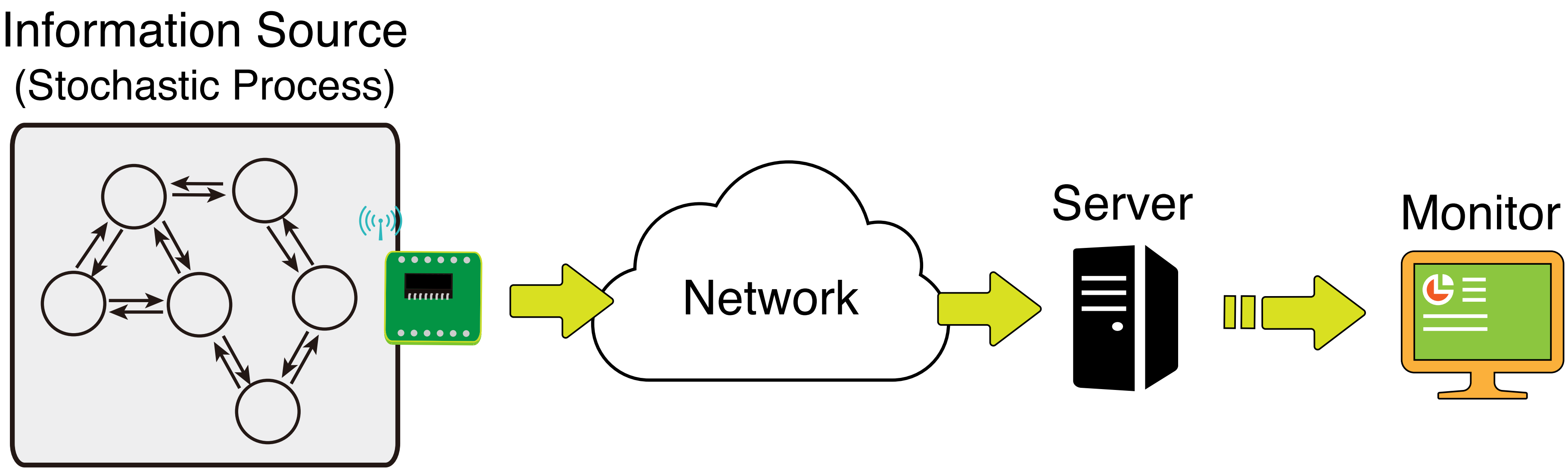
Remote Monitoring System

- The state of an information source is monitored over time



  - ◆ The monitor displays the latest state information received

- A_t: Age of Information (AoI) at time t

$$A_t \triangleq t - \eta_t, \quad t \in \mathbb{R}$$

η_t: Time-stamp of the displayed information at time t

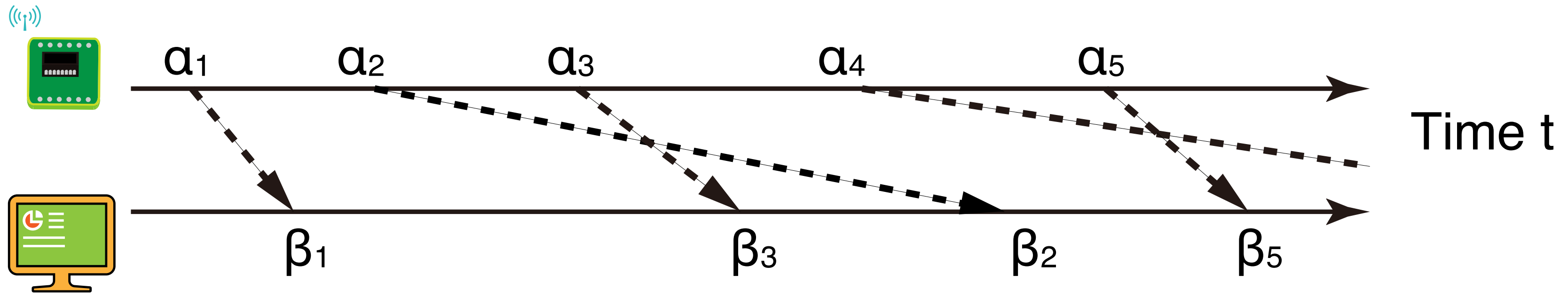
Generic Formulation of AoI (1)

G_n: Inter-generation time of the $(n-1)$st and nth updates

D_n: System delay of the nth update

- nth update is generated at $t = \alpha_n$ and received at $t = \beta_n$

$$\alpha_n = \alpha_{n-1} + G_n, \qquad \beta_n = \alpha_n + D_n$$



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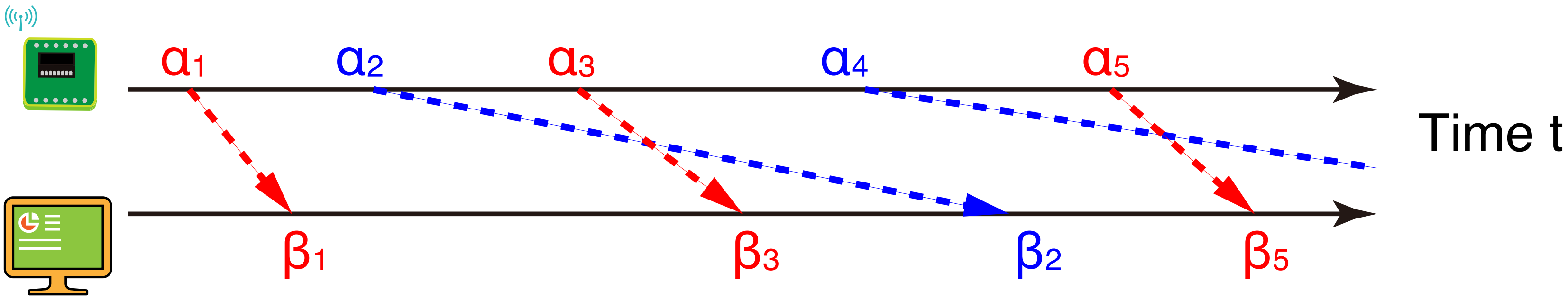
    - Non-overtaken updates are said to be effective



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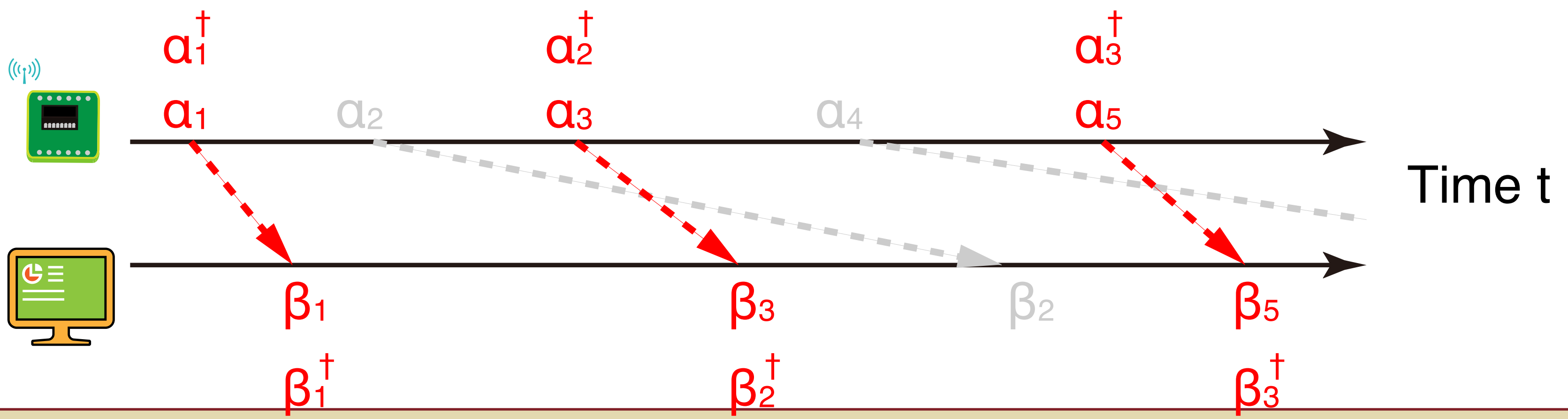

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- AoI process is completely determined in terms of

$$(\alpha_\ell^\dagger)_{\ell\in\mathbb{Z}} := (\alpha_n)_{n\in I} \quad \text{and} \quad (\beta_\ell^\dagger)_{\ell\in\mathbb{Z}} := (\beta_n)_{n\in I}$$

  - Set of effective updates $I := \{n; \beta_n < \min\{\beta_{n+1}, \beta_{n+2}, \dots\}\}$

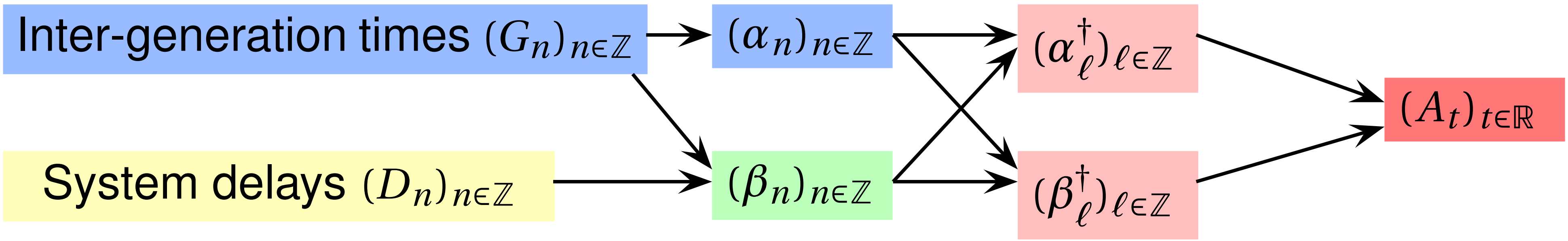
Generic Formulation of AoI (2)

$\alpha_\ell^\dagger$: Generation time of the ℓth effective update

$\beta_\ell^\dagger$: Reception time of the ℓth effective

- AoI A_t at time $t \in \mathbb{R}$ is given by

$$A_t = t - \alpha_\ell^\dagger, \quad \text{for } t \in [\beta_\ell^\dagger, \beta_{\ell+1}^\dagger), \quad \ell \in \mathbb{Z}$$



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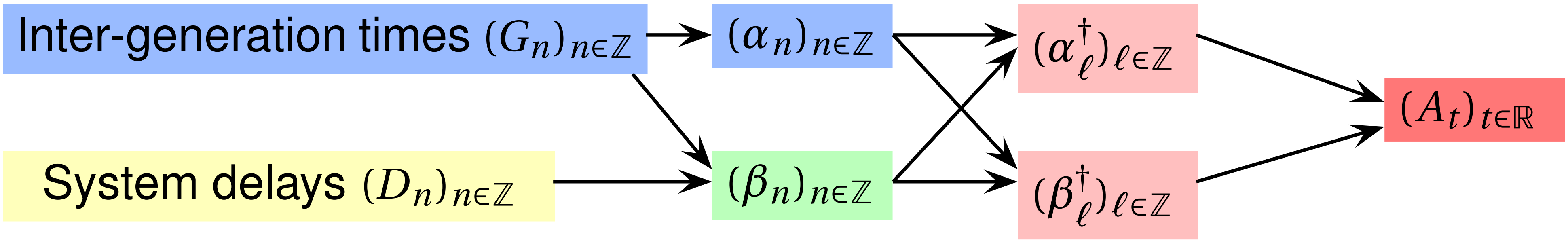

One major direction of AoI researches

Characterize AoI (mean, non-linear cost, distribution, etc.), given statistical properties of $(G_n)_{n\in\mathbb{Z}}$ and $(D_n)_{n\in\mathbb{Z}}$

Queueing Models for AoI Analysis

- $(G_n)_{n\in\mathbb{Z}}$ is usually assumed to be i.i.d. random variables

Example 1) $G_n =$ const. Regular sampling

Example 2) $G_n \sim \mathrm{EXP}(\lambda)$ Poisson sampling

Queueing Models for AoI Analysis

- $(G_n)_{n \in \mathbb{Z}}$ is usually assumed to be i.i.d. random variables
- Modeling frameworks for system delay $(D_n)_{n \in \mathbb{Z}}$
  - Single-server queueing models
    - Formulates congestible resources
    - D_n = [Service time H_n] + [Queueing delay W_n]
    - Smaller $(G_n)_{n \in \mathbb{Z}}$ results in larger $(D_n)_{n \in \mathbb{Z}}$

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    - A model for modern networks and server clusters
    - Equivalent to infinite-server queues

      D_n = [Service time H_n]

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      D_n = [Service time H_n]

Related Works

Mean AoI $\mathrm{E}[A]$ in infinite-server queues has been studied

$$\mathrm{E}[A] = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} A_t \mathrm{d}t$$

- Explicit formula for $\mathrm{E}[A]$ in the M/M/∞ queue[22]
- Model of cloud-gaming, closely related to the D/GI/∞ queue[23]
  - Approximate analysis of $\mathrm{E}[A]$ is given
- Characterization of $\mathrm{E}[A]$ in the GI/GI/∞ queue[24]

[22] C. Kam et al, *IEEE Trans. Inf. Theory*, vol. 65, no. 3, 2016.

[23] R. Yates et al, *Proc. of IEEE INFOCOM 2017*, 2017.

[24] R. Talak et al, *arXiv:1810.04371*, 2018.

Contributions of This Study

G_n: Inter-generation time, D_n: System delay

- We characterize the stationary distribution of the AoI

$$\Pr(A > x) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \mathbb{1}\{A_t > x\}\,\mathrm{d}t$$

under a general settting (GI/GI/∞ with loss):

  - $(G_n)_{n\in\mathbb{Z}}$ and $(D_n)_{n\in\mathbb{Z}}$ follow general distributions
  - $D_n = \infty$ may occur with non-zero probability
    - Corresponding to occurrences of losses

- We further derive specialized results for two selected models:

G_n = const. (D/GI/∞) and $G_n \sim \mathrm{EXP}(\lambda)$ (M/GI/∞)

General Case: GI/GI/∞ Queue with Loss

Model

- $(G_n)_{n\in\mathbb{Z}}$: Sequence of inter-generation times (i.i.d.)
  - Mean generation rate $\lambda := 1/\mathrm{E}[G_n]$
- $(H_n)_{n\in\mathbb{Z}}$: Sequence of system delays (i.i.d. and $D_n = H_n$)
  - H: Generic random variable for system delays
  - CDF $H(x) := \Pr(H \le x)$, CCDF $\overline{H}(x) := 1 - H(x)$
- $H(x)$ may be defective

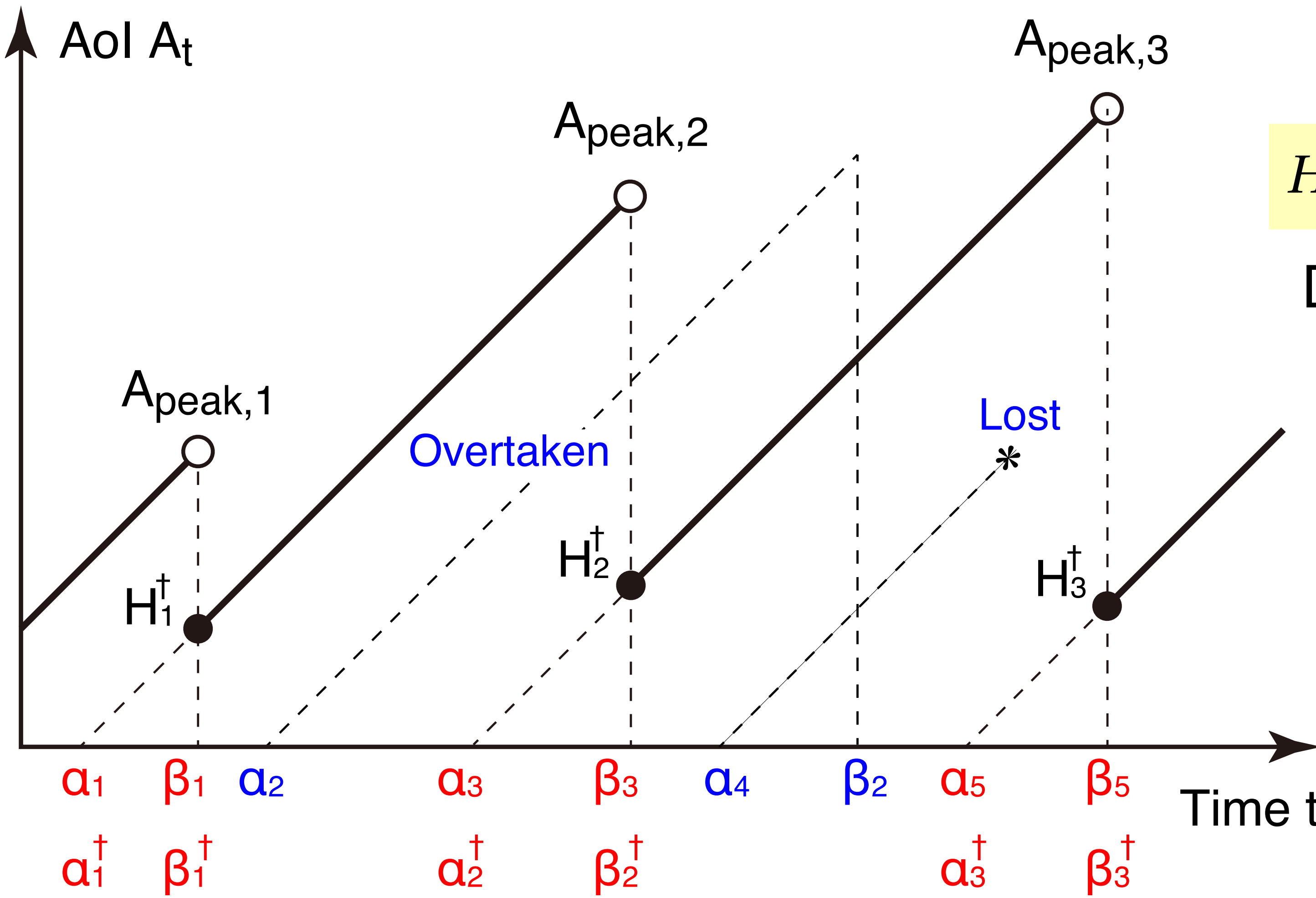
$$\lim_{x\to\infty} \overline{H}(x) = \Pr(H = \infty)$$

(= probability of loss)

Dynamics of AoI in GI/GI/∞

α_n: nth generation time, β_n: nth reception time

$\alpha^\dagger_\ell$: ℓth effective generation time, $\beta^\dagger_\ell$: ℓth effective reception time



$$H^\dagger_\ell := \beta^\dagger_\ell - \alpha^\dagger_\ell$$

Delay of the ℓth effective update

Formula for the AoI Distribution

- We assume that the system is stationary and ergodic
  - A: Stationary AoI, A_{peak}: Stationary peak AoI
  - $H^{\dagger}$: Stationary effective delay

Lemma 1[6].

A has an absolutely continuous distribution with density

$$a(x) = \lambda P_I \cdot \{\Pr(A_{\text{peak}} > x) - \Pr(H^{\dagger} > x)\}$$

P_I: Probability that an update becomes effective

[6] Y. Inoue et al., *IEEE Trans. Inf. Theory*, vol. 65, no. 12, 2019.

AoI Distribution in GI/GI/∞ (1)

- ℓth peak AoI is given by

$$A_{\text{peak},\ell} = H^{\dagger}_{\ell-1} + B^{\dagger}_{\ell}$$

$B^{\dagger}_{\ell} := \beta^{\dagger}_{\ell} - \beta^{\dagger}_{\ell-1}$ Inter-update time

$H^{\dagger}_{\ell} := \beta^{\dagger}_{\ell} - \alpha^{\dagger}_{\ell}$ System delay

AoI Distribution in GI/GI/∞ (1)

- ℓth peak AoI is given by

$$A_{\mathrm{peak},\ell} = H^{\dagger}_{\ell-1} + B^{\dagger}_{\ell}$$

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$H^{\dagger}_{\ell} := \beta^{\dagger}_{\ell} - \alpha^{\dagger}_{\ell}$ System delay

Therefore, we have

$$\Pr(A_{\mathrm{peak},\ell} > x) = \Pr\left(H^{\dagger}_{\ell-1} > x\right) + \Pr\left(H^{\dagger}_{\ell-1} \le x, H^{\dagger}_{\ell-1} + B^{\dagger}_{\ell} > x\right)$$

➨ Substituting this relation into

$$a(x) = \lambda P_I \cdot \left\{\Pr(A_{\mathrm{peak}} > x) - \Pr(H^{\dagger} > x)\right\},$$

we obtain

$$a(x) = \lambda P_I \cdot \Pr\left(H^{\dagger}_{\ell-1} \le x, H^{\dagger}_{\ell-1} + B^{\dagger}_{\ell} > x\right)$$

AoI Distribution in GI/GI/∞ (2)

$$a(x) = \lambda P_I \cdot \Pr\left(H^{\dagger}_{\ell-1} \le x, H^{\dagger}_{\ell-1} + B^{\dagger}_{\ell} > x\right)$$

We can show

$$\Pr\left(H^{\dagger}_{\ell-1} \le x, H^{\dagger}_{\ell-1} + B^{\dagger}_{\ell} > x\right) = \frac{H(x)}{P_I} \mathrm{E}\left[\prod_{n=1}^{\infty} \overline{H}\left(x - \sum_{i=1}^{n} G_i\right)\right],$$

so that we conclude as follows:

Theorem 2. Density function of the AoI is given by

$$a(x) = \lambda H(x) \mathrm{E}\left[\prod_{n=1}^{\infty} \overline{H}\left(x - \sum_{i=1}^{n} G_i\right)\right]$$

Special Cases: D/GI/∞ and M/GI/∞ with Loss

AoI Distribution in D/GI/∞

- Inter-generation times take a constant value τ

$$G_n = \tau, \quad n \in \mathbb{Z}$$

Theorem 3. Let $C_K := [K\tau, (K+1)\tau)$ $(K = 0, 1, \ldots)$

(i) Density function of the AoI in D/GI/∞ is given by

$$a(x) = \frac{H(x)}{\tau} \prod_{n=0}^{K-1} \overline{H}(n\tau + x - K\tau), \quad x \in C_K,$$

(ii) CCDF of the AoI in D/GI/∞ is given by

$$\Pr(A > x) = \frac{1}{\tau} \int_{x-K\tau}^{\tau} \left(\prod_{n=0}^{K-1} \overline{H}(n\tau + u) \right) \mathrm{d}u + \frac{1}{\tau} \int_{0}^{x-K\tau} \left(\prod_{n=0}^{K} \overline{H}(n\tau + u) \right) \mathrm{d}u, \quad x \in C_K$$

AoI Distribution in M/GI/∞

- Inter-generation times follow an exponential distribution

$$\Pr(G_n \le x) = 1 - e^{-\lambda x}$$

Theorem 4.

(i) Density function of the AoI in M/GI/∞ is given by

$$a(x) = \lambda H(x) \exp\left[-\lambda \int_0^x H(y)\mathrm{d}y\right]$$

(ii) CCDF of the AoI in M/GI/∞ is given by

$$\Pr(A > x) = \exp\left[-\lambda \int_0^x H(y)\mathrm{d}y\right]$$

Stochastic Orders[25]

Let X and Y denote non-negative random variables

- **Usual stochastic order $\leq_{\text{st}}$**

$$X \leq_{\text{st}} Y \;\Leftrightarrow\; \Pr(X > x) \leq \Pr(Y > x), \quad \forall x \geq 0$$

  - It is known that $X \leq_{\text{st}} Y$ if and only if $\mathrm{E}[\phi(X)] \leq \mathrm{E}[\phi(Y)]$ holds for all non-decreasing function $\phi(\cdot)$

- **Hazard rate order $\leq_{\text{hr}}$**

$$X \leq_{\text{hr}} Y \;\Leftrightarrow\; [X \mid X > t] \leq_{\text{st}} [Y \mid Y > t], \quad \forall t \geq 0$$

Clearly, we have $X \leq_{\text{hr}} Y \;\Rightarrow\; X \leq_{\text{st}} Y$

[25] M. Shaked & J. Shanthikumar, *Stochastic Orders*, 2006.

Comparison of AoI in D/GI/∞ and M/GI/∞

$$a(x) = \frac{H(x)}{\tau} \prod_{n=0}^{K-1} \overline{H}(n\tau + x - K\tau), \quad x \in [K\tau, (K+1)\tau) \qquad \text{D/GI/}\infty$$

$$a(x) = \lambda H(x) \exp\left[-\lambda \int_0^x H(y)\mathrm{d}y\right], \quad x \geq 0 \qquad \text{M/GI/}\infty$$

- Utilizing these closed-form formulas, we can show

Theorem 13. For D/GI/∞ and M/GI/∞ queues with common generation rate λ and delay distribution $H(\cdot)$,

$$A^{\mathrm{D/GI/}\infty} \leq_{\mathrm{hr}} A^{\mathrm{M/GI/}\infty}$$

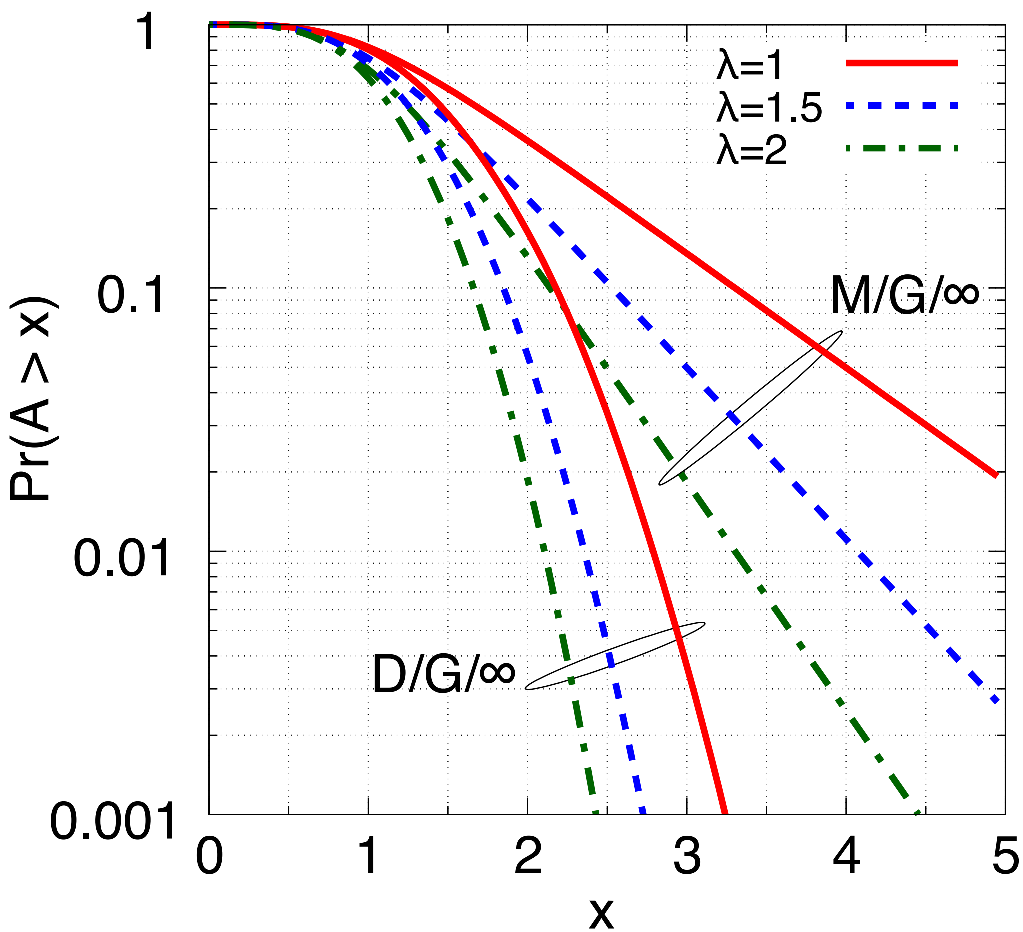
$A^{\mathrm{D/GI/}\infty}$: Stationary AoI in D/GI/∞

$A^{\mathrm{M/GI/}\infty}$: Stationary AoI in M/GI/∞

Numerical Example

Comparison of D/Gamma/∞ and M/Gamma/∞

System delay has $\mathrm{E}[H] = 1$ and $\mathrm{SD}[H] = 0.5$



- Tail probability $\Pr(A > x)$ decays quite faster in D/G/∞ than in M/G/∞
- Regular sampling is more effective than Random sampling

Summary

We considered the AoI distribution in infitite-server queues with loss

- Density function $a(x)$ of the stationary AoI distribution

$$a(x) = \lambda H(x) \mathrm{E}\left[\prod_{n=1}^{\infty} \overline{H}\left(x - \sum_{i=1}^{n} G_i\right)\right], \quad x \geq 0 \qquad \text{GI/GI/}\infty$$

$$a(x) = \frac{H(x)}{\tau} \prod_{n=0}^{K-1} \overline{H}(n\tau + x - K\tau), \quad x \in [K\tau, (K+1)\tau) \qquad \text{D/GI/}\infty$$

$$a(x) = \lambda H(x) \exp\left[-\lambda \int_0^x H(y) \mathrm{d}y\right], \quad x \geq 0 \qquad \text{M/GI/}\infty$$

- Stochastic comparison of the AoI in D/GI/∞ and M/GI/∞

$$A^{\mathrm{D/GI/}\infty} \leq_{\mathrm{hr}} A^{\mathrm{M/GI/}\infty}$$